ABSTRACT OF THE THESIS ENTITLED

A STUDY OF ADDITIVE MAPPINGS IN RINGS WITH INVOLUTION

SUBMITTED FOR THE AWARD OF THE DEGREE OF

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In

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By

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Under the supervision of

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ABSTRACT

The present thesis is a part of the research work carried out by the author during the last four years concerning the study of additive mappings and its various generalizations in the setting of rings with involution. This exposition consists of five chapters and each chapter is subdivided into various sections.

Chapter 1 contains preliminary notions, basic definitions, examples, counter examples and some important well-known results related to our study which may be needed for the development of the subject in subsequent chapters. This chapter is an attempt to make this thesis as self contained as possible. However, the basic knowledge of ring theory has been pre-assumed.

Throughout the discussion all rings are associative unless indicated otherwise and $Z(R)$ denotes the center of the ring R. For elements x and y in a ring R, we shall write $[x, y] = xy - yx$ and $x \circ y = xy + yx$. The element $[x, y]$ is called the Lie product (or the commutator) of elements x and y, and $x \circ y$ is called the Jordan product (or the anti-commutator) of x and y.

Chapter 2 deals with the commutativity of prime rings with involution. An additive mapping $x \mapsto x^*$ on a ring R is called an involution on R if $(xy)^* = y^*x^*$ and $(x^*)^* = x$ holds for all $x, y \in R$. A ring equipped with an involution is called a ring with involution or $*$-ring. Let $d : R \to R$ be an additive mapping such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Then d is said to be a derivation on R. Let S be a nonempty subset of R. A mapping $f : R \to R$ is called centralizing on S if $[f(x), x] \in Z(R)$ for all $x \in S$, and is called commuting on S if $[f(x), x] = 0$ for all $x \in S$. Motivated by the existence of centralizing mappings in rings, the notions of $*$-centralizing and $*$-commuting mappings in rings with involution have been introduced in Section 2.2. Let R be a ring with involution, and S be a nonempty subset of R. A mapping $f : R \to R$ is said to be $*$-centralizing on S if $[f(x), x^*] \in Z(R)$ for all $x \in S$. As a special case, if $[f(x), x^*] = 0$ for all $x \in S$, then f is said to be $*$-commuting on S. A classical result due to Posner [Proc. Amer. Math. Soc. 8 (1957), 1093-1100] states that a prime ring R admitting a nonzero derivation $d : R \to R$ such

that $[d(x), x] \in Z(R)$ for all $x \in R$, must be commutative. The analogous result for centralizing automorphisms on prime rings was obtained by Mayne [Canad. Math. Bull. 19 (1976), 113-117]. Further, this result was subsequently refined and extended by a number of authors in several directions (viz.; [Canad. Math. Bull. 30(1) (1987), 92-101], [Proc. Amer. Math. Soc. 111(2) (1991), 501–510], [Canad. Math. Bull., 24 (1981), 415-421], where further references can be found). In Section 2.2, besides proving some other results, we present a $*$-version of Posner's second theorem mentioned above. It is shown that a prime ring with involution $*$, of characteristic different from two admitting a nonzero derivation $d : R \to R$ such that $[d(x), x^*] \in Z(R)$ for all $x \in R$ and $d(S(R) \cap Z(R)) \neq (0)$, must be commutative. Moreover, Herstein [Canad. Math. Bull. 21(3) (1978), 369–370] proved that a prime ring R of characteristic not two with a nonzero derivation d satisfying $d(x)d(y) = d(y)d(x)$ for all $x, y \in R$, must be commutative. In Section 2.3, we shall continue the similar study in the setting of rings with involution involving derivation. Further, besides studying the Herstein's result [Canad. Math. Bull. 21(3) (1978), 369–370, Theorem 2] in the setting of prime rings with involution, we also explore the commutativity of prime ring with involution $*$, which admits a nonzero derivation d satisfying any one of the following conditions: (i) $d(x) \circ d(x^*) = 0$, (ii) $d([x, x^*]) = 0$, (iii) $d(x \circ x^*) = 0$, (iv) $d([x, x^*]) \pm [x, x^*] = 0$, (v) $d(x \circ x^*) - (x \circ x^*) = 0$, (vi) $d(xx^*) \pm xx^* = 0$, (viii) $d(x)d(x^*) \pm xx^* = 0$ for all $x \in R$. Finally, a counter example has also been given to demonstrate that the restrictions imposed on the hypotheses of the various results are not superfluous.

Chapter 3 is devoted to the study of left (resp. right) centralizers in rings with involution. An additive mapping $T : R \to R$ is said to be a left (resp. right) centralizer of R if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) for all $x, y \in R$. An additive mapping $T : R \to R$ is called a two sided centralizer in case T is a left and a right centralizer of R. The study of such mappings was initiated by Zalar [Comment. Math. Univ. Carol. 32 (1991), 609-614] and subsequently studied by Vukman (see [Comment. Math. Univ. Carol. 38 (1997), 231-240], [Comment. Math. Univ. Carol. 40 (1999), 447-456], [Glas. Math. Ser, III 45(65) (2011), no. 1, 43-53] and the references there in). Over the last two decades, several authors have described the structure of a ring R or the form of the map $T : R \to R$ such that $T(xy) = T(x)y$ (see [Demonstratio Math. 41(4) (2008), 764-771], [Marcel Dekker, Inc. New York. Basel. Hong Kong, (1996), Chapter 2], [Int. J. Open Problems Compt. Math. 3(3) (2010), 267-277] for partial bibliography). In Section 3.2, we study the normality of prime rings involving left centralizers. In fact, it is shown that if a prime ring with involution $*$ of characteristic different from two admits a nonzero left centralizer T such that $[T(x), x^*] = 0$ for all $x \in R$, then R is normal. Further, we also characterizes normal rings and two sided centralizers among all prime rings with involution satisfying certain identities involving left centralizers. Moreover, as an application it is shown that a prime ring with involution of characteristic different from two admitting a nonzero left centralizer T of R such that $[T(x), x^*] = 0$ has the form $T(x) = \lambda x$, where $\lambda \in C$, the extended centriod of R. In [Acta Math. Hungar. 66(4) (1995), 337-343], Bell and Daif proved that if R is a prime ring admitting a nonzero derivation d such that $d([x, y]) = 0$ for all $x, y \in R$, then R is commutative. Further, this result was extended for semiprime ring in [Int. J. Math. & Math. Sci. 21(3) (1998), 471-474]. Recently, Andima and Pajoohesh [Acta Math. Hungar. 128(1-2) (2010), 1-14] proved that an inner derivation satisfying the above mentioned condition on a nonzero ideal of R must be zero on that ideal. In Section 3.3, we generalize the above mentioned result in the setting of prime rings with involution $*$ involving left centralizers and establish the following result: Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. Let T be a left centralizer of R such that $T([x, x^*]) = 0$ for all $x \in R$ and $(S(R) \cap Z(R)) \neq (0)$. Then R is commutative. Similar result is also prove by replacing the Lie product with the Jordan product. In the year 1992, Daif and Bell [Int. J. Math & Math. Sci. 15(1) (1992), 205-206] established that a semiprime ring admitting a derivation d such that $d([x, y]) \pm [x, y] = 0$ for all $x, y \in R$ must be commutative. In [Algebra Colloq. 13(3) (2006), 371–380], Argač generalized this result for a nonzero ideal of a semiprime ring. Recently, Ashraf and Ali [Demonstratio Math. 41(4) (2008), 764-771] studied same result in the setting of prime ring involving left centralizer. In Section 3.4, we study the similar problem in the setting of rings with involution, and consequently it is shown that a prime ring with involution $*$ of characteristic different from two, admits a nonzero left centralizer $T: R \to R$ such that $T([x, x^*]) \pm [x, x^*] = 0$ for all $x \in R$ with $S(R) \cap Z(R) \neq (0)$, must be commutative. Some more results in this direction have also been given.

Material of Chapter 4 concerns with the study of Jordan left $*$-centralizers in rings with involution. An additive mapping $T: R \longrightarrow R$ is said to be a left $*$-centralizer (resp. Jordan left $*$-centralizer) if $T(xy) = T(x)y^*$(resp. $T(x^2) = T(x)x^*$) holds for all $x, y \in R$. The definition of right $*$-centralizer (resp. Jordan right $*$-centralizer) should be self explanatory. An additive mapping $T: R \longrightarrow R$ is said to be a reverse left (resp. right) $*$-centralizer if $T(xy) = T(y)x^*$ (resp. $T(xy) = y^*T(x)$) is fulfilled for all $x, y \in R$. An additive mapping $T: R \longrightarrow R$ is called a $*$-centralizer (resp. reverse $*$-centralizer) if T is both a left and a right $*$-centralizer (resp. reverse left and right $*$-centralizer). For some fixed element $a \in R$, the mapping $x \mapsto ax^*$ is a Jordan left $*$-centralizer and $x \mapsto x^*a$ is a Jordan right $*$-centralizer on R. Clearly, every reverse left $*$-centralizer on a ring R is a Jordan left $*$-centralizer. Thus, it is natural to question that whether the converse of above statement is true. In Section 4.2, it is shown that the answer to this question is affirmative if the underlying $*$-ring R is a 2-torsion free semiprime. Further, we establish a result concerning additive mapping $T: R \to R$ satisfying the relation $T(x^{m+n+1}) = (x^*)^n T(x)(x^*)^m$ for all $x \in R$, where m and n are positive integers. Moreover, some nice characterization of $*$-centralizers in prime and semiprime rings are also given. In particular, it is shown that on a 2-torsion free semiprime ring with involution, any Jordan left $*$-centralizer is of the form $T(x) = qx^*$ for all $x \in R$, where $q \in Q_r(R)$. Further, motivated by the result of Brešar and Vukman [Aequ. Math. 38 (1989), 178-185], we characterize normal rings among all noncommutative prime rings with involution of characteristic different from two by using the theory of Jordan $*$-centralizers. In fact, we prove the following result: Let R be a noncommutative prime ring with involution $*$ such that $char(R) \neq 2$. Then the following conditions are mutually equivalent:

(i) R is normal

(ii) there exists a nonzero commuting Jordan left $*$-centralizer T on R.

In Section 4.3, we study the result obtained by Vukman [Comment. Math. Univ. Carol. 38 (1997), 231-240, Theorem 4] in the setting of rings with involution by replacing left centralizer with Jordan left $*$-centralizer. In fact, we obtain the following result: Let R be a noncommutative 2-torsion free semiprime ring with involution and $S, T : R \longrightarrow R$ be Jordan left $*$-centralizers. Suppose that $[S(x), T(x)]S(x) - S(x)[S(x), T(x)] = 0$ holds for all $x \in R$. Then $[S(x), T(x)] = 0$ for all $x \in R$. Moreover, if R is a prime ring and $S \neq 0$ ($T \neq 0$), then there exists $\lambda \in C$, the extended centroid of R, such that $T = \lambda S$ ($S = \lambda T$). In Section 4.4, we present some applications of our results obtained in previous section.

The last chapter of the thesis deals with the study of certain Jordan $*$-mappings in rings with involution. Let R be a ring with involution $*$, an additive mapping $d : R \to R$ is called a $*$-derivation (resp. Jordan $*$-derivation) if $d(xy) = d(x)y^* + xd(y)$ (resp. $d(x^2) = d(x)x^* + xd(x)$) holds for all $x, y \in R$. Note that the mapping $x \mapsto ax^* - xa$, where a is a fixed element in R, is a Jordan $*$-derivation; such Jordan $*$-derivations are said to be inner. One might expect that any Jordan $*$-derivation on a 2-torsion free (semi)prime $*$-ring is a $*$-derivation, but this is not the case. It is easy to prove that there are no nonzero $*$-derivations on noncommutative prime $*$-rings (see [Aequ. Math. 38 (1989), 178-185] for details). The notion of Jordan $*$-derivations arise naturally in the theory of representability of quadratic functionals with sesquilinear functions (see for example [Colloq. Math. 59(2) (1990), 241–251] and [Studia. Math. 97 (1991), 157-165]). Following [Int. Math. Forum. 13(13-16) (2006), 617–622], an additive mapping $d : R \to R$ is called a Jordan triple $*$-derivation if $d(xyx) = d(x)y^*x^* + xd(y)x^* + xyd(x)$ holds for all $x, y \in R$. One can easily prove that every Jordan $*$-derivation on a 2-torsion free $*$-ring is a Jordan triple $*$-derivation of R. However, the converse of this statement need not be true in general. In [Int. Math. Forum. 13(13-16) (2006), 617–622], Vukman showed that the converse holds if R is 6-torsion free semiprime $*$-ring. Recently, Fošner and Iliševic [Mediterr. J. Math. 5 (2008), 415-427] generalized above mentioned result for 2-torsion free semiprime $*$ rings. Motivated by the above result, we obtain the following result in Section 5.2: Let R be a 2-torsion free semiprime ring with involution $*$, and let $d : R \to R$ be an additive mapping. Suppose that $d(xyx) = d(xy)x^* + xyd(x)$ holds for all pairs $x, y \in R$. In this case d is a $*$-derivation. Moreover, the following result concerning Jordan $*$-derivations has also been given: Let R be a prime ring with involution $*$, of characteristic different from 2. Let d be a nonzero Jordan $*$-derivation of R such that $[d(x), x] = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$. Then R is commutative. Further, motivated by the definition of $*$-derivation and Jordan $*$-derivation, we introduce the notion of left $*$-derivation and Jordan left $*$-derivation and obtain similar results in the setting of these mappings. In Section 5.3, we introduce the notion of symmetric Jordan $*$-biderivation and symmetric Jordan triple $*$-biderivation as follows: a symmetric biadditive map $D : R \times R \to R$ is said to be a symmetric Jordan $*$-biderivation if $D(x^2, z) = D(x, z)x^* + xD(x, z)$ holds for all $x, z \in R$. A symmetric biadditive map $D : R \times R \to R$ is called a symmetric Jordan triple $*$-biderivation if $D(xyx, z) = D(x, z)y^*x^* + xD(y, z)x^* + xyD(x, z)$ holds for all $x, y, z \in R$. It is obvious to see that every symmetric Jordan $*$-biderivation on 2-torsion free ring with involution is a symmetric Jordan triple $*$-biderivation. But the converse need not be true in general. In the present section, we establish a set of conditions under which every symmetric Jordan triple $*$-biderivation on a ring with involution is a symmetric Jordan $*$-biderivation. More precisely, we prove that on a 2-torsion free semiprime ring with involution $*$, every symmetric Jordan triple $*$-biderivation is a symmetric Jordan $*$-biderivation. Section 5.4 is devoted to the study of the applications of our results obtained in previous section. In fact, besides proving some other results, it is shown that in a $(m+n)!$-torsion free noncommutative prime ring with involution $*$, which admit Jordan $*$-derivations d, g such that either $d(x^m)x^n - x^n g(x^m) = 0$ for all $x \in R$ or $d(x^m)x^n + x^n g(x^m) = 0$ for all $x \in R$, where m, n are non-negative integers. Moreover, if $S(R) \cap Z(R) \neq (0)$, then $d = g = 0$.

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This Thesis Is Dedicated

To

My Father

ABDUL GANI DAR

All I have and will accomplish

are only possible due to his love and sacrifices

Candidate's Declaration

I, Nadeem Ahmad Dar, the Department of Mathematics, Aligarh Muslim University, Aligarh certify that the work embodied in this Ph. D. thesis is my own bonafide work carried out by me under the supervision of Dr. Shakir Ali at the Department of Mathematics, Aligarh Muslim University, Aligarh. The matter embodied in this Ph. D. thesis has not been submitted for the award of any other degree.

I declare that I have faithfully acknowledged, given credit to and referred to the research workers wherever their works have been cited in the text and the body of the thesis. I further certify that I have not willfully lifted up some other's work, paragraph, text, data, result *etc.* reported in the journals, books, magazines, reports, dissertations, thesis *etc.* or available at web-sites and included them in this Ph. D. thesis and cited as my own work.

Date: 24.07.2014

(Nadeem Ahmad Dar)

....

Certificate from the Supervisor

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

Signature of the Supervisor:..

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Certificate

This is to certify that the thesis entitled **"A study of additive mappings in rings with involution"** is based on a part of the research work of **Mr. Nadeem Ahmad Dar** carried out under my guidance in the Department of Mathematics, Aligarh Muslim University, Aligarh. To the best of my knowledge, the work included in the thesis is original and has not been submitted to any other university or institution for the award of the degree.

It is further certified that **Mr. Nadeem Ahmad Dar** has fulfilled the prescribed conditions of duration and nature given in the statutes and ordinances of the Aligarh Muslim University, Aligarh.

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Course/Comprehensive Examination/Pre-Submission Seminar Completion Certificate

This is to certify that Mr. Nadeem Ahmad Dar, the Department of Mathematics, Aligarh Muslim University, Aligarh has satisfactorily completed the course work/comprehensive examination and pre-submission seminar requirement which is part of his Ph. D. programme.

24/7

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Table of Contents

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July 2014 (Nadeem Ahmad Dar)

Preface

Besides homomorphism, anti-homomorphism and isomorphism etcetera, the additive maps like centralizer and derivation have also started attracting a large number of algebraists just after the work's of Posner [115] and Zalar [146]. In the year 1957, Posner [115] obtained two very striking results which state namely; (i) In a prime ring of characteristic different from two, if the iterate of two derivations is a derivation, then at least one of them must be zero (ii) A prime ring R admitting a nonzero derivation d such that for every ring element x, the commutator $[x, d(x)] \in Z(R)$, the center of R, must be commutative. An additive mapping $d : R \to R$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $T : R \to R$ is called a left (resp. right) centralizer of R if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) holds for all $x, y \in R$. An additive mapping T is called a centralizer in case T is both a left and a right centralizer of R. During the last two decades, there has been ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of additive maps like centralizers and derivations of R. Recently, many authors (see for instance [2, 6, 7, 11, 15, 27, 34, 41, 60, 78, 104], where further references can be found) have obtained commutativity of prime and semiprime rings with centralizers or derivations satisfying certain polynomial constraints. The present thesis entitled "**A study of additive mappings in rings with involution**" contains similar work carried out by the author during last four years concerning derivations, Jordan $*$-derivations, left centralizers, left $*$-centralizers, Jordan left $*$-centralizers etcetera in the setting of prime and semiprime rings with involution at the Department of Mathematics, Aligarh Muslim University, Aligarh.

This exposition consists of five chapters and each chapter is subdivided into various sections. The definitions, examples, counter examples, remarks, results etcetera have been specified with the double decimal numbers. The first figure denotes the number of the definition, the example, counter examples or the result as the case may be in a particular chapter. For example, Theorem 3.2.1 refers to the first theorem appearing in the second section of the third chapter.

Chapter 1 contains preliminary notions, basic definitions and some important known results which may be needed for the development of the subject in the subsequent chapters. The aim of this chapter is to make this thesis as self contained as possible. However, the basic knowledge of ring theory has been pre-assumed.

Chapter 2 deals with the commutativity of prime rings with involution. An additive mapping $x \mapsto x^*$ on a ring R is called an involution on R if $(xy)^* = y^*x^*$ and $(x^*)^* = x$ holds for all $x, y \in R$. A ring equipped with an involution is called a ring with involution or $*$-ring. Let $d : R \to R$ be an additive mapping such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Then d is said to be a derivation on R. Let S be a nonempty subset of R. A mapping $f : R \to R$ is called centralizing on S if $[f(x), x] \in Z(R)$ for all $x \in S$, and is called commuting on S if $[f(x), x] = 0$ for all $x \in S$. Motivated by the existence of centralizing mappings in rings, the notions of $*$-centralizing and $*$-commuting mappings in rings with involution have been introduced in Section 2.2. Let R be a ring with involution, and S be a nonempty subset of R. A mapping $f : R \to R$ is said to be $*$-centralizing on S if $[f(x), x^*] \in Z(R)$ for all $x \in S$. As a special case, if $[f(x), x^*] = 0$ for all $x \in S$, then f is said to be $*$-commuting on S. A classical result due to Posner [115] states that a prime ring R admitting a nonzero derivation $d : R \to R$ such that $[d(x), x] \in Z(R)$ for all $x \in R$, must be commutative. The analogous result for centralizing automorphisms on prime rings was obtained by Mayne [103]. Further, this result was subsequently refined and extended by a number of authors in several directions (viz.; [21, 23, 34, 35, 40, 43, 52, 54], where further references can be found). In Section 2.2, besides proving some other results, we present a $*$-version of Posner's second theorem mentioned above. It is shown that a prime ring with involution $*$, of characteristic different from two admitting a nonzero derivation $d : R \to R$ such that $[d(x), x^*] \in Z(R)$ for all $x \in R$ and $d(S(R) \cap Z(R)) \neq (0)$, must be commutative. Moreover, Herstein [76] proved that a prime ring R of characteristic not two with a nonzero derivation d satisfying $d(x)d(y) = d(y)d(x)$ for all $x, y \in R$, must be commutative. In Section 2.3, we shall continue the similar study in the setting of rings with involution involving derivation. Further, besides studying the Herstein's result [76, Theorem 2] in the setting of prime rings with involution, we also explore the commutativity of prime ring with involution $*$, which admits a nonzero derivation d

satisfying any one of the following conditions: (i) $d(x) \circ d(x^*) = 0$, (ii) $d([x, x^*]) = 0$, (iii) $d(x \circ x^*) = 0$, (iv) $d([x, x^*]) \pm [x, x^*] = 0$, (v) $d(x \circ x^*) - (x \circ x^*) = 0$, (vi) $d(xx^*) \pm xx^* = 0$, (viii) $d(x)d(x^*) \pm xx^* = 0$ for all $x \in R$. Finally, a counter example has also been given to demonstrate that the restrictions imposed on the hypotheses of the various results are not superfluous.

Chapter 3 is devoted to the study of left (resp. right) centralizers in rings with involution. An additive mapping $T : R \to R$ is said to be a left (resp. right) centralizer of R if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) for all $x, y \in R$. An additive mapping $T : R \to R$ is called a two sided centralizer in case T is a left and a right centralizer of R. The study of such mappings was initiated by Zalar [146] and subsequently studied by Vukman (see [131, 132, 133, 137] and the references there in). Over the last two decades, several authors have described the structure of a ring R or the form of the map $T : R \to R$ such that $T(xy) = T(x)y$ (see [15], [37, Chapter 2], [109] for partial bibliography). In Section 3.2, we study the normality of prime rings involving left centralizers. In fact, it is shown that if a prime ring with involution $*$ of characteristic different from two admits a nonzero left centralizer T such that $[T(x), x^*] = 0$ for all $x \in R$, then R is normal. Further, we also characterizes normal rings and two sided centralizers among all prime rings with involution satisfying certain identities involving left centralizers. Moreover, as an application it is shown that a prime ring with involution of characteristic different from two admitting a nonzero left centralizer T of R such that $[T(x), x^*] = 0$ has the form $T(x) = \lambda x$, where $\lambda \in C$, the extended centriod of R. In [34], Bell and Daif proved that if R is a prime ring admitting a nonzero derivation d such that $d([x, y]) = 0$ for all $x, y \in R$, then R is commutative. Further, this result was extended for semiprime ring in [54]. Recently, Andima and Pajoohesh [11] proved that an inner derivation satisfying the above mentioned condition on a nonzero ideal of R must be zero on that ideal. In Section 3.3, we generalize the above mentioned result in the setting of prime rings with involution $*$ involving left centralizers and establish the following result: Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. Let T be a left centralizer of R such that $T([x, x^*]) = 0$ for all $x \in R$ and $(S(R) \cap Z(R)) \neq (0)$. Then R is commutative. Similar result is also prove by replacing the Lie product with the Jordan product. In the year 1992, Daif and Bell [55] established that a semiprime ring admitting a derivation d such that $d([x, y]) \pm [x, y] = 0$ for all $x, y \in R$ must be commutative. In [14], Argač generalized this result for a nonzero ideal of a semiprime ring. Recently, Ashraf and Ali [15] studied same result in the setting of prime ring involving left centralizer. In Section 3.4, we study the similar problem in the setting of rings with involution, and consequently it is shown that a prime ring with involution $*$ of characteristic different from two, admits a nonzero left centralizer $T : R \to R$ such that $T([x, x^*]) \pm [x, x^*] = 0$ for all $x \in R$ with $S(R) \cap Z(R) \neq (0)$, must be commutative. Some more results in this direction have also been given.

Material of Chapter 4 concerns with the study of Jordan left $*$-centralizers in rings with involution. An additive mapping $T : R \longrightarrow R$ is said to be a left $*$-centralizer (resp. Jordan left $*$-centralizer) if $T(xy) = T(x)y^*$(resp. $T(x^2) = T(x)x^*$) holds for all $x, y \in R$. The definition of right $*$-centralizer (resp. Jordan right $*$-centralizer) should be self explanatory. An additive mapping $T : R \longrightarrow R$ is said to be a reverse left (resp. right) $*$-centralizer if $T(xy) = T(y)x^*$ (resp. $T(xy) = y^*T(x)$) is fulfilled for all $x, y \in R$. An additive mapping $T : R \longrightarrow R$ is called a $*$-centralizer (resp. reverse $*$-centralizer) if T is both a left and a right $*$-centralizer (resp. reverse left and right $*$-centralizer). For some fixed element $a \in R$, the mapping $x \mapsto ax^*$ is a Jordan left $*$-centralizer and $x \mapsto x^*a$ is a Jordan right $*$-centralizer on R. Clearly, every reverse left $*$-centralizer on a ring R is a Jordan left $*$-centralizer. Thus, it is natural to question that whether the converse of above statement is true. In Section 4.2, it is shown that the answer to this question is affirmative if the underlying $*$-ring R is a 2-torsion free semiprime. Further, we establish a result concerning additive mapping $T : R \to R$ satisfying the relation $T(x^{m+n+1}) = (x^*)^n T(x)(x^*)^m$ for all $x \in R$, where m and n are positive integers. Moreover, some nice characterization of $*$-centralizers in prime and semiprime rings are also given. In particular, it is shown that on a 2-torsion free semiprime ring with involution, any Jordan left $*$-centralizer is of the form $T(x) = qx^*$ for all $x \in R$, where $q \in Q_r(R)$. Further, motivated by the result of Brešar and Vukman [45], we characterize normal rings among all noncommutative prime rings with involution of characteristic different from two by using the theory of Jordan $*$-centralizers. In fact, we prove the following result: Let R be a noncommutative prime ring with involution $*$ such that $char(R) \neq 2$. Then the following conditions are mutually equivalent:

(i) R is normal

(ii) there exists a nonzero commuting Jordan left $*$-centralizer T on R.

In Section 4.3, we study the result obtained by Vukman [131, Theorem 4] in the setting of rings with involution by replacing left centralizer with Jordan left $*$-centralizer. In fact, we obtain the following result: Let R be a noncommutative 2-torsion free semiprime ring with involution and $S, T : R \longrightarrow R$ be Jordan left $*$-centralizers. Suppose that $[S(x), T(x)]S(x) - S(x)[S(x), T(x)] = 0$ holds for all $x \in R$. Then $[S(x), T(x)] = 0$ for all $x \in R$. Moreover, if R is a prime ring and $S \neq 0$ ($T \neq 0$), then there exists $\lambda \in C$, the extended centroid of R, such that $T = \lambda S$ ($S = \lambda T$). In Section 4.4, we present some applications of our results obtained in previous section.

The last chapter of the thesis deals with the study of certain Jordan $*$-mappings in rings with involution. Let R be a ring with involution $*$, an additive mapping $d : R \rightarrow R$ is called a $*$-derivation (resp. Jordan $*$-derivation) if $d(xy) = d(x)y^* + xd(y)$ (resp. $d(x^2) = d(x)x^* + xd(x)$) holds for all $x, y \in R$. Note that the mapping $x \mapsto ax^* - xa$, where a is a fixed element in R, is a Jordan $*$-derivation; such Jordan $*$-derivations are said to be inner. One might expect that any Jordan $*$-derivation on a 2-torsion free (semi)prime $*$-ring is a $*$-derivation, but this is not the case. It is easy to prove that there are no nonzero $*$-derivations on noncommutative prime $*$-rings (see [45] for details). The notion of Jordan $*$-derivations arise naturally in the theory of representability of quadratic functionals with sesquilinear functions (see for example [117] and [118]). Following [135], an additive mapping $d : R \rightarrow R$ is called a Jordan triple $*$-derivation if $d(xyx) = d(x)y^*x^* + xd(y)x^* + xyd(x)$ holds for all $x, y \in R$. One can easily prove that every Jordan $*$-derivation on a 2-torsion free $*$-ring is a Jordan triple $*$-derivation of R. However, the converse of this statement need not be true in general. In [135], Vukman showed that the converse holds if R is 6-torsion free semiprime $*$-ring. Recently, Fošner and Iliševic [61] generalized above mentioned result for 2-torsion free semiprime $*$ rings. Motivated by the above result, we obtain the following result in Section 5.2: Let R be a 2-torsion free semiprime ring with involution $*$, and let $d : R \rightarrow R$ be an additive mapping. Suppose that $d(xyx) = d(xy)x^* + xyd(x)$ holds for all pairs $x, y \in R$. In this case d is a $*$-derivation. Moreover, the following result concerning Jordan $*$-derivations has also been given: Let R be a prime ring with involution $*$, of characteristic different from 2. Let d be a nonzero Jordan $*$-

derivation of R such that $[d(x), x] = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$. Then R is commutative. Further, motivated by the definition of $*$-derivation and Jordan $*$-derivation, we introduce the notion of left $*$-derivation and Jordan left $*$-derivation and obtain similar results in the setting of these mappings. In Section 5.3, we introduce the notion of symmetric Jordan $*$-biderivation and symmetric Jordan triple $*$-biderivation as follows: a symmetric biadditive map $D : R \times R \to R$ is said to be a symmetric Jordan $*$-biderivation if $D(x^2, z) = D(x, z)x^* + xD(x, z)$ holds for all $x, z \in R$. A symmetric biadditive map $D : R \times R \to R$ is called a symmetric Jordan triple $*$-biderivation if $D(xyx, z) = D(x, z)y^*x^* + xD(y, z)x^* + xyD(x, z)$ holds for all $x, y, z \in R$. It is obvious to see that every symmetric Jordan $*$-biderivation on 2-torsion free ring with involution is a symmetric Jordan triple $*$-biderivation. But the converse need not be true in general. In the present section, we establish a set of conditions under which every symmetric Jordan triple $*$-biderivation on a ring with involution is a symmetric Jordan $*$-biderivation. More precisely, we prove that on a 2-torsion free semiprime ring with involution $*$, every symmetric Jordan triple $*$-biderivation is a symmetric Jordan $*$-biderivation. Section 5.4 is devoted to the study of the applications of our results obtained in previous section. In fact, besides proving some other results, it is shown that in a $(m + n)!$-torsion free noncommutative prime ring with involution $*$, which admit Jordan $*$-derivations d, g such that either $d(x^m)x^n - x^n g(x^m) = 0$ for all $x \in R$ or $d(x^m)x^n + x^n g(x^m) = 0$ for all $x \in R$, where m, n are non-negative integers. Moreover, if $S(R) \cap Z(R) \neq (0)$, then $d = g = 0$.

An extensive bibliography of the existing literature related to the subject matter is also included after Chapter 5.

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CHAPTER-1

Some Preliminaries

Chapter 1

Some Preliminaries

1.1 Introduction

The purpose of present chapter is to introduce some basic definitions, preliminary notions and some results which we shall be using in the subsequent chapters of our thesis. The knowledge of some elementary concepts like groups, rings, ideals, fields, modules, homomorphism etcetera have been preassumed. Throughout the thesis, unless otherwise mentioned, R will denote an associative ring (may be without unity) containing at least two elements. For most of the material included in this chapter, we refer to Beidar et al. [37], Herstein [73], [75], Jacobson [82] and McCoy [105].

1.2 Some definitions and examples

In the present section we give a brief exposition of some important terminology in the theory of rings and algebras. Examples and counter examples are also included in this section to make the matter presented in the section self explanatory and to give a clear sketch of the various notions. We start our discussion with the following definition:

Definition 1.2.1 (Prime ideal). A proper ideal P of R is called a *prime ideal* of R if for any two ideals A and B of R, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

Remark 1.2.1. If P is an ideal in a ring R, then the following conditions are equivalent:

(i) P is a prime ideal of R.

(ii) If $a, b \in P$ such that $aRb \subseteq P$, then $a \in P$ or $b \in P$.

(iii) If (a) and (b) are principal ideals in R such that $(a)(b) \subseteq P$, then $a \in P$ or $b \in P$.

(*iv*) If U and V are left (right) ideals in R such that $UV \subseteq P$, then $U \subseteq P$ or $V \subseteq P$.

Definition 1.2.2 (Prime ring). A ring R is said to be a *prime ring* if and only if the zero ideal is a prime ideal in R.

Remark 1.2.2. Equivalently, a ring R is a prime ring if and only if any one of the following holds:

(*i*) If A and B are ideals in R such that AB=(0), then A=(0) or B=(0).

(*ii*) If $a, b \in R$ such that $aRb = (0)$, then $a = 0$ or $b = 0$.

Definition 1.2.3 (Semiprime ideal). An ideal P in a ring R is said to be a *semiprime ideal* in R if for every ideal I of R, $I^2 \subseteq P$ implies $I \subseteq P$.

Remark 1.2.3. (*i*) A prime ideal is necessarily semiprime, but the converse need not be true in general.
(*ii*) Intersection of prime (semiprime) ideals is semiprime. Thus, in the ring $\mathbb{Z}$ of integers, ideal $(2) \cap (3) = (6)$ is semiprime which is not prime.

Definition 1.2.4 (Semiprime ring). A ring R which has no nonzero nilpotent ideal is said to be a *semiprime ring.*

Remark 1.2.4. A ring R is semiprime if and only if for any $a \in R$, $aRa = (0)$ implies that $a = 0$.

Definition 1.2.5 (Dense right (left) ideal). A right (resp. left) ideal I of R is said to be *dense right* (resp. *left*) *ideal* of R if for any $0 \neq r_1 \in R$, $r_2 \in R$ there exists $r \in R$ such that $r_1 r \neq 0$ and $r_2 r \in I$ (resp. $rr_1 \neq 0$ and $rr_2 \in I$).

The collection of all dense right ideal of R will be denoted by $D(R)$.

Remark 1.2.5. Let I, J, $S \in D(R)$ and let $f : I \to R$ be homomorphism of right R-modules. Then

(*i*) $f^{-1}(J) = \{a \in I \mid f(a) \in J\} \in D(R)$.

(*ii*) $I \cap J \in D(R)$.

(*iii*) $IJ \in D(R)$.

Definition 1.2.6 (Maximal right ring of quotients). Let R be a semiprime ring, $\Im$ be the set of all pairs (U, f) where $U \neq (0)$ is a dense right ideal of R and $f : U \to R$ is a right R-module map of U into R. Define a relation '$\sim$' on $\Im$ such that $(U, f) \sim (V, g)$ if $f = g$ on some dense right ideal $W \neq (0)$ of R, where $W \subseteq U \cap V$. It can be easily check that $\sim$ is an equivalence relation on $\Im$. Let $Q_r(R)$ be the set of equivalence classes of $\Im$. Denote the equivalence class determined by (U, f) as $\tilde{f}$. For $\tilde{f} = cl(U, f)$, $\tilde{g} = cl(V, g) \in Q_r(R)$, define addition and multiplication on $Q_r(R)$ as $\tilde{f}+\tilde{g} = cl(U\cap V,\ f+g)$ and $\tilde{f}.\tilde{g} = cl(g^{-1}(U),\ fg)$. Thus $Q_r(R)$ forms an associative ring with identity relative to above defined operations known as *maximal right ring of quotients* or *right Utumi quotient ring of R*.

Remark 1.2.6. Let R be a semiprime ring. Then $Q_r(R)$ satisfies:

(i) R is a subring of $Q_r(R)$.

(ii) For all $q \in Q_r(R)$ there exists $I \in D(R)$ such that $qI \subseteq R$.

(iii) For all $q \in Q_r(R)$ and $I \in D(R)$, $qI = 0$ if and only if $q = 0$.

(iv) For all $I \in D(R)$ and $f : I_R \to R_R$ there exists $q \in Q_r(R)$ such that $f(x) = qx$ for all $x \in I$.

Furthermore, properties (i)-(iv) characterize ring $Q_r(R)$ up to isomorphism.

Definition 1.2.7 (Symmetric ring of quotients). Let R be a semiprime ring and $\mathbb{I} = I(R) = \{I \mid I \textit{ is an ideal of } R \textit{ and } l(I) = 0\}$. We note that $\mathbb{I}$ is closed under products and finite intersections. Then the symmetric ring of quotients of R denoted by $Q_s(R)$ is defined as follows:

$$Q_s(R) = \{q \in Q_r(R) \mid qJ \cup Jq \subseteq R \textit{ for some } J \in \mathbb{I}\}.$$

Definition 1.2.8 (Center of ring). The center of a ring R is the set of all those elements of R which commute with every element of R and is denoted as $Z(R)$ i.e., $Z(R) = \{x \in R \mid xr = rx \text{ for all } r \in R\}$.

Thus, a ring R is commutative if and only if $Z(R) = R$.

Remark 1.2.7. (i) The center of a prime ring is free from zero divisors.

(*ii*) The center of a semiprime ring contains no nonzero nilpotent element.

Definition 1.2.9 (Extended centroid). The center C of $Q_r(R)$ is known as extended centroid of R.

Remark 1.2.8. If R is a prime ring, then extended centroid of R is a field.

Definition 1.2.10 (Central closure). Let R be a semiprime ring. Then the subring RC of $Q_r(R)$ is said to be the *central closure* of R. Further, R is called *centrally closed* if it coincides with its central closure i.e., $R = RC$.

Definition 1.2.11 (Characteristic of a ring). Let R be a ring. If there exists a positive integer n such that $nx = 0$ for all $x \in R$, then the smallest positive integer with this property is called the characteristic of the ring R and is denoted by char(R). If no such positive integer exists, then R is said to be of characteristic zero.

Definition 1.2.12 (Torsion free element). An element $x \in R$ is called n-torsion free if $nx = 0$ implies $x = 0$.

If $nx = 0$ implies $x = 0$ for all $x \in R$, we say the ring R is n-torsion free.

Definition 1.2.13 (Lie and Jordan Structures). Let R be a ring. Then using its operations, two new products can be induced as follows:

(*i*) for all $x, y \in R$, the *Lie product* $[x, y] = xy - yx$,

(*ii*) for all $x, y \in R$, the *Jordan product* $x \circ y = xy + yx$.

Remark 1.2.9. For any $x, y, z \in R$, the following identities are obvious,

(*i*) $[xy, z] = x[y, z] + [x, z]y$,

(*ii*) $[x, yz] = [x, y]z + y[x, z]$,

(*iii*) $[[x, y], z]] + [[y, z], x]] + [[z, x], y]] = 0$, (Jacobi's Identity)

(*iv*) $x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$,

(*v*) $(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$.

Definition 1.2.14 (Lie (Jordan) subring). A nonempty subset A of R is said to be a *Lie* (resp. *Jordan*) *subring* of R if A is an additive subgroup of R and for any $a, b \in A$, implies that $[a, b]$ (resp. $(a \circ b)$) is also in A.

Definition 1.2.15 (Lie (Jordan) ideal). A nonempty subset U of R is said to be a *Lie* (resp. *Jordan*) *ideal* of R if U is an additive subgroup of R and whenever $u \in U$ and $r \in R$, then $[u, r] \in U$ (resp. $(u \circ r) \in U$).

Definition 1.2.16 (Derivation and Jordan derivation). An additive mapping $d : R \to R$ is said to be a *derivation* (resp. *Jordan derivation*) on R if $d(xy) = d(x)y + xd(y)$ (resp. $d(x^2) = d(x)x + xd(x)$) holds for all $x, y \in R$.

Example 1.2.1. The most natural example of a non trivial derivation is the usual differentiation on the ring $F[x]$ of polynomials defined over a field F.

Definition 1.2.17 (Inner derivation). For a fixed $a \in R$, define $d_a : R \to R$ such that $d_a(x) = [a, x]$ for all $x \in R$. Then d is called an *inner derivation* of R associated with 'a' and usually denoted by I_a.

It is obvious to see that every inner derivation on a ring R is a derivation. But the converse need not be true in general.

Example 1.2.2. Let $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{Z} \right\}$. Define a mapping $d : R \to R$ as follows:

$$d \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for all } a, b, c \in \mathbb{Z}.$$

It can be easily seen that d is a derivation on R which is not an inner derivation on R.

Remark 1.2.10. If d is a derivation on R and $r \in Z(R)$, then $d(r) \in Z(R)$.

Definition 1.2.18 (Jordan triple derivation). An additive mapping $d : R \to R$ is said to be a *Jordan triple derivation* on R if $d(xyx) = d(x)yx + xd(y)x + xyd(x)$ holds for all $x, y \in R$.

Definition 1.2.19 (Left derivation). An additive mapping $d : R \to R$ is called a left derivation of R if $d(xy) = xd(y) + yd(x)$ for all $x, y \in R$.

Example 1.2.3. Let $\mathbb{S}$ be any ring. Next let $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{S} \right\}$. Define a mapping $d : R \to R$ as follows:

$$d \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ for all } a, b, c \in \mathbb{S}.$$

It can be easily seen that d is a left derivation on R.

Definition 1.2.20 (Jordan Left derivation). An additive mapping $d : R \to R$ is called a Jordan left derivation of R if $d(x^2) = 2xd(x)$ for all $x \in R$.

Remark 1.2.11. It is easy to see that every left derivation on a ring R is a Jordan left derivation. However, in general a Jordan left derivation need not be a left derivation.

Example 1.2.4. Let R be a commutative ring and let $a \in R$ such that $xax = 0$ for all $x \in R$ but $xay \neq 0$, for some x and y, $x \neq y$. define a map $d : R \to R$ as follows:

$$d(x) = xa + ax.$$

Then, d is a Jordan left derivation but not a left derivation.

Definition 1.2.21 (Centralizer). An additive mapping $T : R \to R$ is called a left (resp. right) centralizer if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) holds for all $x, y \in R$. Also, T is a *centralizer* if it is both a left as well as a right centralizer.

Remark 1.2.12. (*i*) If T is a centralizer on a semiprime ring R, then there exists an element $\lambda \in C$, the extended centroid of R such that $T(x) = \lambda x$ for all $x \in R$.

(*ii*) If R is a ring with identity, then T is a left centralizer of R if and only if $T(x) = ax$ for all $x \in R$ and some fixed element a of R.

Definition 1.2.22 (Generalized derivation). An additive mapping $F : R \to R$ is said to be a *generalized derivation* on R if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$.

Definition 1.2.23 (Generalized Jordan derivation). An additive mapping $F : R \to R$ is said to be a *generalized Jordan derivation* on R if there exists a Jordan derivation $d : R \to R$ such that $F(x^2) = F(x)x + xd(x)$ for all $x \in R$.

Definition 1.2.24 (Generalized Jordan triple derivation). An additive mapping $F : R \to R$ is said to be a *generalized Jordan triple derivation* on R if there exists a Jordan triple derivation $d : R \to R$ such that $F(xyx) = F(x)yx + xd(y)x + xyd(x)$ for all $x, y \in R$.

Clearly, generalized derivation covers the concept of derivation and left centralizer. If we take $F = d$, then generalized derivation becomes a derivation and if we take $d = 0$ then it becomes a left centralizer. By the following example, we can easily see that every generalized derivation need not be a derivation.

Example 1.2.5. Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$. Define $F, d : R \to R$ such that

$$F\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \text{ and } d\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix} \text{ for all } a, b, c \in \mathbb{Z}.$$

Then, F is a generalized derivation of R with associated derivation d, but not a derivation of R.

Definition 1.2.25 (Involution). An involution on a ring R is a map $* : R \longrightarrow R$ satisfying the following conditions:

(*i*) $(x + y)^* = x^* + y^*$,

(*ii*) $(xy)^* = y^* x^*$,

(*iii*) $(x^*)^* = x$ for all $x, y \in R$.

A ring equipped with an involution is called a $*$-ring or ring with involution.

Example 1.2.6. Let $M_{n\times n}(\mathbb{R})$ be the set of all $n \times n$ matrices over the real field $\mathbb{R}$. Take $A \in M_{n\times n}$ and define $* : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ such that $A^* = A^T$, the transpose of A. Then $*$ is an involution on $M_{n\times n}(\mathbb{R})$ and hence $M_{n\times n}(\mathbb{R})$ is a ring with involution.

Definition 1.2.26 (Symmetric element). An element x of a $*$-ring R is said to be *symmetric* if $x^* = x$.

Definition 1.2.27 (Skew symmetric element). An element x of a $*$-ring R is said to be *skew symmetric* if $x^* = -x$.

The set of all *symmetric* and *skew-symmetric* elements of R are denoted by $H(R)$ and $S(R)$, respectively. Therefore we have

$$H(R) = \{x \in R \mid x^* = x\} \text{ and } S(R) = \{x \in R \mid x^* = -x\}.$$

Definition 1.2.28 (Normal ring). A ring R with involution $*$ is said to be *normal* if $xx^* = x^*x$ for all $x \in R$. Equivalently, R is said to be *normal* if $hk = kh$ for all $h \in H(R)$ and $k \in S(R)$.

Example 1.2.7. Let R be the ring of real quaternions. Then the mapping $* : R \to R$ defined by $x^* = \bar{x}$, where $\bar{x}$ denotes the conjugate of x, is an involution on R and $xx^* = x^*x$ for all $x \in R$. That is, R is normal.

Remark 1.2.13. (*i*) Let R be a ring with involution $*$. The set $H(R)$ of symmetric elements of R form a Jordan subring of R and the set $S(R)$ of skew symmetric elements of R form a Lie ideal of R

(*ii*) The involution is said to be of first kind if $Z(R) \subseteq H(R)$, otherwise it is said to be of the second kind. In the latter case $S(R) \cap Z(R) \neq (0)$.

(*iii*) If R is a 2-torsion free $*$-ring, then every element $x \in R$ can be uniquely represented in the form $2x = h + k$, where $h \in H(R)$ and $k \in S(R)$.

(*iv*) Let R be a simple ring with involution $*$ of characteristic different from two. In this case, since $2R$ is an ideal of R, so must be R, for any $x \in R$ $\frac{x}{2}$ makes sense and so, since

$$x = \frac{x + x^*}{2} + \frac{x - x^*}{2}, \ R = H(R) + S(R), \ H(R) \cap S(R) = (0).$$

1.3 Some known results

In this section we write some well-know results which will be used in the subsequent chapters.

Lemma 1.3.1. [21, Theorem 4.3] *Let R be a 2-torsion free prime ring and I a nonzero ideal of R. If R admits a nonzero derivation d such that $d(x) \circ d(y) = 0$ for all $x, y \in R$, then R is commutative.*

Lemma 1.3.2. [37, Theorem 6.4.6] *Let R be a prime ring with extended centroid C, an anti-automorphism g and $Q = Q_r(R)$. Suppose that*

$$0 \neq \phi = \phi(x_1, ..., x_n, x_1^g, ..., x_n^g) \in Q_c < X \cup X^g >$$

is a g-identity on $0 \neq K \lhd R$. Then ϕ is a g-identity of $Q_s = Q_s(R)$.

Lemma 1.3.3. [41, Proposition 3.1] *Let R be a 2-torsion free semiprime ring and U be a Jordan subring of R. If an additive mapping F of R into itself is centralizing on U, then F is commuting on U.*

Lemma 1.3.4. *[42, Theorem 2] Let R be a 2-torsion free semiprime ring. If an additive mapping $f : R \to R$ is skew-commuting on R, then $f = 0$.*

Lemma 1.3.5. [43, Corollary 4.2] *Let R be a semiprime ring and let $f : R \to R$ be a commuting additive mapping. Then there exists $\lambda \in C$ and an additive mapping $\mu : R \to C$ such that $f(x) = \lambda x + \mu(x)$ for all $x \in R$.*

Lemma 1.3.6. [52, Lemma 1] *Let R be an $m!$-torsion free ring. Suppose $y_1, y_2, ..., y_m \in R$ satisfying $\alpha y_1 + \alpha^2 y_2 + ... + \alpha^m y_m = 0$ for $\alpha = 1, 2, ..., m$. Then $y_i = 0$ for all i.*

Lemma 1.3.7. [61, Theorem 5.4] *Let R be a 2-torsion free semiprime $*$-ring and let $E, \delta : R \to R$ be additive mappings. Then the following conditions are mutually equivalent:*

(a) for all $x, y \in R$,

$$E(xyx) = E(x)y^*x^* + x\delta(y)x^* + xy\delta(x),$$
$$\delta(xyx) = \delta(x)y^*x^* + x\delta(y)x^* + xy\delta(x);$$

(b) for all $x \in R$,

$$E(x^2) = E(x)x^* + x\delta(x),$$
$$\delta(x^2) = \delta(x)x^* + x\delta(x).$$

Lemma 1.3.8. [63, Theorem 3] *Let m and n be positive integers, and let R be a prime ring with $char(R) = 0$ or $m+n+1 \leq char(R)$. Let $T : R \to R$ be an additive mapping satisfying the relation $T(x^{m+n+1}) = (x)^m T(x)(x)^n$ for all $x \in R$. Then T is a two sided centralizer.*

Lemma 1.3.9. [75, Theorem 2.1.2] *Let R be any 2-torsion free semiprime ring. Suppose that A is both a subring of R and a Lie ideal of R. If A is commutative, then if $a \in A$, we must have $a^2 \in Z(R)$.*

Lemma 1.3.10 ([75], pp. 20-23). *Suppose that the elements a_i, b_i in the central closure of a prime ring R satisfy $\Sigma a_i x b_i = 0$ for all $x \in R$. If $b_i \neq 0$ for some i, then $a_i's$ are C-dependent.*

Lemma 1.3.11. [77, Theorem] *Let R be a prime ring and d be a nonzero derivation of R such that $ad(x) = d(x)a$ for all $x \in R$. Then $a = 0$.*

Lemma 1.3.12. [104, Lemma 4] *Let b and ab be in the center of a prime ring R. If $b \neq 0$, then $a \in Z(R)$, the center of R.*

Lemma 1.3.13. [115, Lemma 1] *Let d be a derivation of a prime ring R and a be an element of R. If $ad(x) = 0$ (or $d(x)a = 0$) for all $x \in R$, then either $a = 0$ or $d = 0$.*

Lemma 1.3.14. [115, Lemma 3] *Let R be a prime ring, and d a nonzero derivation of R such that $[d(x), x] = 0$ for all $x \in R$. Then, R is commutative.*

Lemma 1.3.15. [134, Lemma 3] *Let R be a semiprime ring and let $f : R \to R$ be an additive mapping. If either $f(x)x = 0$ or $xf(x) = 0$ holds for all $x \in R$, then $f = 0$.*

Lemma 1.3.16. [146, Proposition 1.4] *Let R be a 2-torsion free semiprime ring and $T : R \to R$ an additive mapping which satisfies $T(x^2) = T(x)x$ for all $x \in R$. Then, T is a left centralizer.*

CHAPTER-2

On Commutativity of Prime Rings with Involution

Chapter 2

On Commutativity of Prime Rings with Involution

2.1 Introduction

In the early stages of general ring theory, striking success of that theory were theorems which asserted the commutativity of the ring when the elements of a ring were subjected to certain types of algebraic conditions. Much of the initial thrust of the work in this area was either authored by Herstein or inspired by his work (viz.; [67, 68, 69, 70, 72, 74]). The other significant contributors in this direction have been Ashraf, Bell, Hirano, Kezlan, Komatsu, Tominga, Yaqub with a variety of coauthors. A good cross-section of such results, and the techniques needed to obtain them, can be found in [1, 16, 17, 18, 19, 28, 29, 30, 31, 32, 65, 66, 84, 86, 96, 145], where further references can be found. Later as the theory evolved, many authors investigated the relationship between the commutativity of the ring R and certain special types of maps on R. In this direction the concept of centralizing and commuting maps is of great importance. A mapping f of R into itself is called centralizing if $[f(x), x] \in Z(R)$ holds for all $x \in R$; in the special case when $[f(x), x] = 0$ holds for all $x \in R$, the mapping f is said to be commuting. The first result in this direction is due to Divinsky [58], who proved that a simple artinian ring is commutative if it has a commuting non-trivial automorphism. Two years later, Posner [115] established that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Over the last few decades, several authors have subsequently refined

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and extended these results in various directions and have established the relationship between the commutativity of a ring R and the existence of certain specific additive maps like derivations, centralizers, generalized derivations and automorphisms of R (viz.; [20, 21, 22, 23, 34, 35, 38, 39, 76, 79] where further references can be found). In [90], Lee and Lee considered Posner's result mentioned above when the ring R is equipped with involution (for symmetric ~~or~~ skew symmetric elements) and consequently provided counter examples that one cannot expect to conclude the commutativity of R even if R is assumed to be a division ring. For instance, in the ring of quaternions, if $*$ is the usual conjugation $(\alpha+\beta i+\gamma J+\delta k)^* = \alpha-\beta i-\gamma J-\delta k$, all symmetric elements are central and hence the property $[d(x), x] \in Z(R)$ holds trivially for all symmetric elements x. Thus, the usual ring theoretic methods devised earlier are not adequate to handle these new situations. Therefore, it is interesting to study the earlier results on rings in the setting of rings with involution involving some additive mappings.

Section 2.2 opens with the definition of $*$-centralizing (resp. $*$-commuting) mapping in rings with involution and subsequently characterization of additive mappings which are $*$-commuting on semiprime rings with involution are given. Further, besides presenting a $*$-version of Posner's second theorem [115], we describe the structure of a pair of additive mappings which are $*$-commuting on semiprime ring with involution. In fact, the following result has been obtained: Let R be a 2-torsion free semiprime ring with involution $*$ and let f, g be the additive mappings of R commuting with $*$ such that $f(x)x^* - x^*g(x) = 0$ for all $x \in R$, then there exists $\lambda \in C$ and an additive mapping $\mu : R \to C$ such that $f(x) = g(x) = \lambda x^* + \mu'(x)$ for all $x \in R$, where $\mu'(x) = \mu(x^*)$.

Section 2.3 is devoted to the study of commutativity of rings with involution involving derivations. We study the Herstein's result [76, Theorem 2] in the setting of prime rings with involution. Moreover, we explore the commutativity of prime ring with involution $*$ which admits a derivation d satisfying any one of the following conditions: (i) $d(x)\circ d(x^*) = 0$, (ii) $d([x, x^*]) = 0$, (iii) $d(x \circ x^*) = 0$, (iv) $d([x, x^*]) \pm [x, x^*] = 0$, (v) $d(x \circ x^*) - (x \circ x^*) = 0$, (vi) $d(xx^*) \pm xx^* = 0$, (vii) $d(x)d(x^*) \pm xx^* = 0$ for all $x \in R$.

2.2 On $*$-commuting and $*$-centralizing mappings in rings with involution

With a view to make our text self contained, we begin with the following definition.

Definition 2.2.1. *Let R be a ring and S be a nonempty subset of R. A mapping $f : R \to R$ is said to be centralizing on S if $[f(x), x] \in Z(R)$ for all $x \in S$. As a special case, if $[f(x), x] = 0$ for all $x \in S$, then f is said to be commuting on S.*

Motivated by the definition of centralizing and commuting mappings, we introduce the notion of $*$-centralizing and $*$-commuting mappings as follows:

Definition 2.2.2. *Let R be a ring with involution $*$ and S be a nonempty subset of R. A mapping $f : R \to R$ is said to be $*$-centralizing on S if $[f(x), x^*] \in Z(R)$ for all $x \in S$. As a special case, if $[f(x), x^*] = 0$ for all $x \in S$, then f is said to be $*$-commuting on S.*

Definition 2.2.3. *Let R be a ring with involution $*$ and S be a nonempty subset of R. A mapping $f : R \to R$ is said to be skew $*$-centralizing on S if $f(x) \circ x^* \in Z(R)$ for all $x \in S$. As a special case, if $f(x) \circ x^* = 0$ for all $x \in S$, then f is said to be skew $*$-commuting on S.*

Notice that for any central element a, the map $x \mapsto ax^*$ is $*$-commuting and $*$-centralizing but neither commuting nor centralizing on R. Thus, it is reasonable to study the behaviour of such mappings in the setting of prime and semiprime rings with involution.

Over the last three decades, several authors have investigated the relationship between the commutativity of a ring and the existence of certain specific types of maps on R. The first result in this direction is due to Divinsky [58], who proved that a simple artinian ring is commutative if it has a commuting non-trivial automorphism. This result was subsequently refined and extended by a number of authors in various directions (see for example [13, 96, 120], where further references can be looked). In the year 1957, Posner [115] proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Besides proving some other results on prime and semiprime rings with involution, the main objective of this section is to present a $*$-version of Posner's theorem [115, Theorem 2]. The result which we want to refer states as follows:

Theorem 2.2.1. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If d is a nonzero derivation of R such that $[d(x), x^*] \in Z(R)$ for all $x \in R$ and $d(S(R) \cap Z(R)) \neq (0)$, then R is commutative.*

In order to develop the proof of the above theorem, we begin with following:

Lemma 2.2.1. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If $S(R) \cap Z(R) \neq (0)$ and R is normal, then R is commutative.*

Proof. By the hypothesis, we have R is normal, that is $hk = kh$ *for all* $h \in H(R)$ *and* $k \in S(R)$. Let x be an arbitrary element of R. Then $x - x^* \in S(R)$, hence

$$h(x - x^*) = (x - x^*)h \text{ for all } x \in R \text{ and } h \in H(R). \tag{2.2.1}$$

Also for $s \in S(R) \cap Z(R)$, $s(x + x^*) \in S(R)$ for all $x \in R$. This gives $hs(x + x^*) = s(x + x^*)h$ for all $x \in R$ and $h \in H(R)$. This can be further written as $s(h(x+x^*) - (x+x^*)h) = 0$ for all $x \in R$, $h \in H(R)$. Since the centre of a prime ring is free from zero divisors, we have either $s = 0$ or $h(x + x^*) - (x + x^*)h = 0$ for all $x \in R$ and $h \in H(R)$. Since $S(R) \cap Z(R) \neq (0)$, from the last expression we conclude that

$$h(x + x^*) = (x + x^*)h \text{ for all } x \in R \text{ and } h \in H(R). \tag{2.2.2}$$

Adding (2.2.1) and (2.2.2), we get $2hx = 2xh$ for all $x \in R$ and $h \in H(R)$. Since $char(R) \neq 2$, we obtain $hx = xh$ *for all* $x \in R$ *and* $h \in H(R)$. Since $x + x^* \in H(R)$, the last relation yields that

$$(x + x^*)y = y(x + x^*) \text{ for all } x, y \in R. \tag{2.2.3}$$

Again since for $s \in S(R) \cap Z(R)$, $s(x - x^*) \in H(R)$, we conclude that $s((x - x^*)y - y(x - x^*)) = 0$ *for all* $x, y \in R$. Using the fact that the centre of a prime ring is free from zero divisors, we get $s = 0$ or $(x - x^*)y = y(x - x^*)$ for all $x, y \in R$. But $S(R) \cap Z(R) \neq (0)$, so we are forced to conclude that

$$(x - x^*)y = y(x - x^*) \text{ for all } x, y \in R. \tag{2.2.4}$$

Adding (2.2.3) and (2.2.4), we obtain $2xy = 2yx$. Since $char(R) \neq 2$, the last expression yields that $xy = yx$ for all $x, y \in R$. This proves R is commutative. □

Proposition 2.2.1. *Let R be a 2-torsion free semiprime ring with involution $*$. If an* *additive mapping f of R into itself is such that $[f(x), x^*] \in Z(R)$ for all $x \in R$, then $[f(x), x^*] = 0$ for all $x \in R$.*

Proof. We have $[f(x), x^*] \in Z(R)$ for all $x \in R$. Replacing x by x^*, we obtain $[f(x^*), x] \in Z(R)$ for all $x \in R$. Define a new map $g : R \to R$ such that $g(x) = f(x^*)$ for all $x \in R$. Then it is easy to verify that g is additive and hence $[g(x), x] \in Z(R)$ for all $x \in R$. In view of Lemma 1.3.3, we conclude that $[g(x), x] = 0$ for all $x \in R$. This implies that $[f(x^*), x] = 0$ for all $x \in R$ and hence $[f(x), x^*] = 0$ for all $x \in R$. The proposition is now proved. □

Theorem 2.2.2. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If d is a nonzero derivation of R such that $[d(x), x^*] = 0$ for all $x \in R$ and $d(Z(R)) \neq (0)$, then R is normal.*

Proof. By the assumption, we have

$$[d(x), x^*] = 0 \text{ for all } x \in R. \tag{2.2.5}$$

A lineralization of (2.2.5) yields that

$$[d(x), y^*] + [d(y), x^*] = 0 \text{ for all } x, y \in R.$$

Which can be further written as

$$[d(x), y^*] = [x^*, d(y)] \text{ for all } x, y \in R. \tag{2.2.6}$$

On the one hand, we have

$$\begin{aligned} [d(x), (y^*)^2] &= [x^*, d(y^2)] \\ &= [x^*, d(y)y + yd(y)] \\ &= [x^*, d(y)]y + d(y)[x^*, y] + [x^*, y]d(y) + y[x^*, d(y)] \end{aligned} \tag{2.2.7}$$

for all $x, y \in R$. On the other hand, we have

$$\begin{aligned} [d(x), (y^*)^2] &= [d(x), y^*]y^* + y^*[d(x), y^*] \\ &= [x^*, d(y)]y^* + y^*[x^*, d(y)] \text{ for all } x, y \in R. \end{aligned} \tag{2.2.8}$$

On comparing (2.2.7) and (2.2.8), we find that

$$[x^*, d(y)](y^* - y) + (y^* - y)[x^*, d(y)] = d(y)[x^*, y] + [x^*, y]d(y) \tag{2.2.9}$$

for all $x, y \in R$. Taking $y = x$ in (2.2.9), we get

$$d(x)[x^*, x] + [x^*, x]d(x) = 0 \text{ for all } x \in R.$$

Replacing x by $h + k$ where $h \in H(R)$ and $k \in S(R)$, we obtain

$$\begin{aligned} 0 &= d(h)[h,k] - d(h)[k,h] + d(k)[h,k] - d(k)[k,h] + [h,k]d(h) \\ &- [k,h]d(h) + [h,k]d(k) - [k,h]d(k) \end{aligned}$$

for all $h \in H(R)$ and $k \in S(R)$. Since $char(R) \neq 2$, the last relation reduces to

$$d(h)[h,k] + d(k)[h,k] + [h,k]d(h) + [h,k]d(k) = 0 \tag{2.2.10}$$

for all $h \in H(R)$ and $k \in S(R)$. Replacing h by $-h$, we have

$$d(h)[h,k] - d(k)[h,k] + [h,k]d(h) - [h,k]d(k) = 0 \tag{2.2.11}$$

for all $h \in H(R)$ *and* $k \in S(R)$. Adding (2.2.10) and (2.2.11) and using the fact that $char(R) \neq 2$, we get

$$d(h)[h,k] + [h,k]d(h) = 0 \text{ for all } h \in H(R) \text{ and } k \in S(R). \tag{2.2.12}$$

Now substituting h for y in (2.2.9), we get

$$d(h)[x^*, h] + [x^*, h]d(h) = 0 \text{ for all } x \in R \text{ and } h \in H(R).$$

Replacing x by x^*, we obtain

$$d(h)[x, h] + [x, h]d(h) = 0 \text{ for all } x \in R \text{ and } h \in H(R). \tag{2.2.13}$$

If $h \in Z(R)$, $d(h) \in Z(R)$ and hence $d(h)^2 \in Z(R)$. If $h \notin Z(R)$, then the map defined by $D(x) = [x, h]$ for all $x \in R$ is a nonzero derivation and by (2.2.13) $d(h)^2 D(x) =$

$D(x)d(h)^2$; thus, in view of the Lemma 1.3.11, we conclude that $d(h)^2 \in Z(R)$. We now have

$$d(h)^2 \in Z(R) \text{ for all } h \in H(R). \tag{2.2.14}$$

Since $d(Z(R)) \neq (0)$, we must have $d(Z(R) \cap H(R)) \neq (0)$ or $d(Z(R) \cap S(R)) \neq (0)$. Moreover, if $z \in Z(R) \cap S(R)$ and $d(z) \neq 0$, then $z^2 \in Z(R) \cap H(R)$ and $d(z^2) = 2zd(z) \neq 0$; consequently, $d(Z(R) \cap H(R)) \neq (0)$. Let $0 \neq h_0 \in Z(R) \cap H(R)$ such that $d(h_0) \neq 0$, and let h be an arbitrary element of $H(R)$. Then by (2.2.14) $d(h)^2, d(h_0)^2$ and $d(h + h_0)^2 = d(h)^2 + d(h_0)^2 + 2d(h_0)d(h)$ are all in $Z(R)$; and it follows that $2d(h_0)d(h) \in Z(R)$ and hence $d(h) \in Z(R)$. Thus

$$d(h) \in Z(R) \text{ for all } h \in H(R). \tag{2.2.15}$$

It now follows from (2.2.12) that if $h \in H(R)$ and $d(h) \neq 0$, then $[h, k] = 0$ for all $k \in S(R)$. If $h \in H(R)$ and $d(h) = 0$, we choose $h_1 \in H(R)$ such that $d(h_1) \neq 0$; and since $d(h_1 + h) \neq 0$, we have $[h_1 + h, k] = 0$ and hence $[h, k] = 0$ for all $k \in S(R)$. We have shown that $[h, k] = 0$ for all $h \in H(R)$ and $k \in S(R)$, hence R is normal. Thus the theorem is proved. □

Now we are well equipped to furnish the proof of Theorem 2.2.1.

Proof of Theorem 2.2.1. By the hypothesis, we have $[d(x), x^*] \in Z(R)$ for all $x \in R$. Application of Proposition 2.2.1 yields that $[d(x), x^*] = 0$ for all $x \in R$. In view of Theorem 2.2.2, we conclude that R is normal. Hence, R is commutative by Lemma 2.2.1. This completes the proof of the theorem.

In [41], Brešar proved that if the additive mapping $f : R \to R$ is centralizing on a prime ring of characteristic different from two, then there exists $\lambda \in C$ and an additive mapping $\mu : R \to C$ such that $f(x) = \lambda x + \mu(x)$ for all $x \in R$. Later in [43], he extended this result in the setting of semiprime rings and proved the following result:

Theorem 2.2.3. *Let R be a 2-torsion free semiprime ring and let $f : R \to R$ be a centralizing additive mapping. Then there exists $\lambda \in C$ and an additive mapping $\mu : R \to C$ such that $f(x) = \lambda x + \mu(x)$ for all $x \in R$.*

Motivated by the above mentioned result, we study the similar problem for $*$-centralizing mappings in semiprime rings with involution. Precisely, we prove the following:

Theorem 2.2.4. *Let R be a 2-torsion free semiprime ring with involution $*$. If an additive mapping $f : R \to R$ is $*$-centralizing on R, then there exists $\lambda \in C$ and an additive mapping $\mu : R \to C$ such that $f(x) = \lambda x^* + \mu'(x)$ for all $x \in R$, where $\mu'(x) = \mu(x^*)$.*

Proof. Since R is a semiprime ring with involution and $f : R \to R$ is $*$-centralizing on R. So by Proposition 2.2.1 we have $[f(x), x^*] = 0$ for all $x \in R$. Replacing x by x^*, we obtain $[f(x^*), x] = 0$ for all $x \in R$. Let $g : R \to R$ be defined by $g(x) = f(x^*)$ for all $x \in R$. Then it is easy to verify that g is additive and therefore we have $[g(x), x] = 0$ for all $x \in R$. In view of Lemma 1.3.5, we conclude that $g(x) = \lambda x + \mu(x)$ *for all* $x \in R$ and hence $f(x^*) = \lambda x + \mu(x)$ where $\mu : R \longrightarrow C$ and $\lambda \in C$. Replacing x by x^* in the last expression, we obtain $f(x) = \lambda x^* + \mu(x^*)$ for all $x \in R$. This implies $f(x) = \lambda x^* + \mu'(x)$ for all $x \in R$, where $\mu'(x) = \mu(x^*)$. This completes the proof. $\square$

As an immediate consequence of Theorem 2.2.4, we have the following result:

Corollary 2.2.1. *Let R be a 2-torsion free semiprime ring with involution $*$. If an additive mapping $f : R \to R$ is $*$-commuting on R, then there exists $\lambda \in C$ and an additive mapping $\mu : R \to C$ such that $f(x) = \lambda x^* + \mu'(x)$ for all $x \in R$, where $\mu'(x) = \mu(x^*)$.*

Using the similar approach as we have used in Theorem 2.2.4 and making use of Lemma 1.3.4 instead of Lemma 1.3.5, we have the following result.

Theorem 2.2.5. *Let R be a 2-torsion free semiprime ring with involution $*$. If an additive mapping $f : R \to R$ is such that $f(x)x^* + x^*f(x) = 0$ for all $x \in R$, then $f = 0$.*

In the next theorem, we characterize a pair of additive mappings which are $*$-commuting on a semiprime ring with involution. In fact, we prove the following result:

Theorem 2.2.6. *Let R be a 2-torsion free semiprime ring with involution $*$. Next, let f, g be the additive mappings of R and commuting with $*$ such that $f(x)x^* - x^*g(x) = 0$ for all $x \in R$. Then there exists $\lambda \in C$ and an additive mapping $\mu : R \to C$ such that $f(x) = g(x) = \lambda x^* + \mu'(x)$ for all $x \in R$, where $\mu'(x) = \mu(x^*)$.*

Proof. Suppose $f, g : R \to R$ be additive mappings such that $f(x)x^* - x^*g(x) = 0$ for all $x \in R$. We use the fact that R being a semiprime ring, the intersection of all prime ideals in R is zero. Let P be the prime ideal such that the quotient ring R/P is of characteristic not two. Since P being a prime ideal, the quotient ring R/P is a prime ring. Moreover, if $*$ is the involution on R, the involution on R/P is defined as $(x+P)^* = x^* + P$ for all $x \in R$. Now by the given hypothesis, we have

$$f(x)x^* - x^*g(x) = 0 \text{ for all } x \in R. \tag{2.2.16}$$

Replacing x by x^* in (2.2.16), we have

$$f(x^*)x - xg(x^*) = 0 \text{ for all } x \in R. \tag{2.2.17}$$

Linearizing (2.2.16), we get

$$f(x)y^* + f(y)x^* - x^*g(y) - y^*g(x) = 0 \text{ for all } x, y \in R.$$

Replacing y by p^*, where $p \in P$, in the above equation, we get $f(p^*)x^* - x^*g(p^*) \in P$ for all $p \in P$ and $x \in R$. Replacing x by x^*, we get

$$f(p^*)x - xg(p^*) \in P \text{ for all } p \in P \text{ and } x \in R. \tag{2.2.18}$$

In particular, $f(p^*)xy - xyg(p^*) \in P$ that is, $(f(p^*)x - xg(p^*))y - x(yg(p^*) - g(p^*)y) \in P$ for all $p \in P$ and $x \in R$. The first term is contained in P because of (2.2.18), hence $x(yg(p^*) - g(p^*)y) \in P$ for all $p \in P$ and $x, y \in R$. P being the prime ideal of R, we have

$$yg(p^*) - g(p^*)y \in P \text{ for all } p \in P \text{ and } y \in R. \tag{2.2.19}$$

Combining (2.2.18) and (2.2.19), we obtain

$$f(p^*)x - g(p^*)x \in P \text{ for all } p \in P \text{ and } x \in R.$$

Let $h = f - g$, then we have $h(p^*)x \in P$ for all $p \in P$ and $x \in R$. Since P is a prime ideal of R, we have $h(p^*) \in P$ for all $p \in P$. Therefore h will induce an additive mapping F on R/P, defined by $F(x+P) = h(x^*) + P$ for all $x \in R$. Since both f and g commute with $*$ and making use of equation (2.2.17), it is easy to prove that F is skew-commuting on R/P. Therefore in view of Lemma 1.3.4, $F = 0$. This implies that $h(x) \in P$ for all $x \in R$. That is, we have proved that the range of h is contained in any prime ideal P such that R/P is of characteristic different from two. We show that the intersection of all such prime ideal is zero. Now since R is a semiprime ring, there exists a family of prime ideals $\{P_a | a \in \Lambda\}$ such that $\bigcap_a P_a = 0$. Let $B = \{b \in \Lambda |$ the characteristic of R/P_b is not $2\}$ and $C = \{c \in \Lambda |$ the characteristic of R/P_c is $2\}$. Then $2x \in \bigcap_c P_c$ for every $x \in R$. Therefore given any $x \in \bigcap_b P_b$, we have $2x \in (\bigcap_c P_c) \cap (\bigcap_b P_b) = \bigcap_a P_a = 0$, and so $x = 0$, since R is 2-torsion free. Thus $\bigcap_b P_b = 0$. This proves that $h(x) = 0$ that is, $f(x) = g(x)$ for all $x \in R$. This gives because of (2.2.16) $g(x)x^* - x^*g(x) = 0$ for all $x \in R$. Therefore in view of Theorem 2.2.4, we obtain $f(x) = g(x) = \lambda x^* + \mu'(x)$ for all $x \in R$, where $\mu'(x) = \mu(x^*)$. This completes the proof of the theorem. □

2.3 On derivations in prime rings with involution

In [76], Herstein proved that a prime ring R of characteristic not two with a nonzero derivation d satisfying $d(x)d(y) = d(y)d(x)$ for all $x, y \in R$, must be commutative. Further, Daif [54] showed that if a 2-torsion free semiprime ring R admits a derivation d such that $d(x)d(y) = d(y)d(x)$ for all $x, y \in I$, where I is a nonzero ideal of R and d is nonzero on I, then R contains a nonzero central ideal. Further this result was extended by many authors (viz.; [7, 21, 78], where further refrences can be found). In view of Herstein's result [76, Theorem 2] mentioned above, it is a natural question: What can we say about the commutativity of a prime ring if we replace y by x^* in the above mentioned condition ? In this direction, we have succeeded in establishing the following result:

Theorem 2.3.1. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If d is a nonzero derivation of R such that $[d(x), d(x^*)] = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$, then R is commutative.*

Proof. By the assumption, we have

$$[d(x), d(x^*)] = 0 \text{ for all } x \in R. \tag{2.3.1}$$

A lineralization of (2.3.1) yields that

$$[d(x), d(y^*)] + [d(y), d(x^*)] = 0 \text{ for all } x, y \in R. \tag{2.3.2}$$

Replacing y by xx^* in (2.3.2), we arrive at

$$\begin{aligned}
0 &= [d(x), d(x)x^* + xd(x^*)] + [d(x)x^* + xd(x^*), d(x^*)] \\
&= d(x)[d(x), x^*] + [d(x), x]d(x^*) + x[d(x), d(x^*)] \\
&+ [d(x), d(x^*)]x^* + d(x)[x^*, d(x^*)] + [x, d(x^*)]d(x^*) \\
&= d(x)[d(x), x^*] + [d(x), x]d(x^*) + d(x)[x^*, d(x^*)] + [x, d(x^*)]d(x^*)
\end{aligned}$$

for all $x \in R$. That is,

$$d(x)[d(x), x^*] + [d(x), x]d(x^*) + d(x)[x^*, d(x^*)] + [x, d(x^*)]d(x^*) = 0 \tag{2.3.3}$$

for all $x \in R$. Replacing x by $x + h'$, where $h' \in H(R) \cap Z(R)$, we obtain

$$d(h')[d(x), x^*] + [d(x), x]d(h') + d(h')[x^*, d(x^*)] + [x, d(x^*)]d(h') = 0.$$

This can be further written as

$$d(h')([d(x), x^*] + [d(x), x] + [x^*, d(x^*)] + [x, d(x^*)]) = 0$$

for all $h' \in H(R) \cap Z(R)$ and $x \in R$. Since the centre of a prime ring is free from zero divisors we get either $d(h') = 0$ for all $h' \in H(R) \cap Z(R)$ or $[d(x), x^*] + [d(x), x] + [x^*, d(x^*)] + [x, d(x^*)] = 0$ for all $x \in R$. Suppose

$$d(h') = 0 \text{ for all } h' \in H(R) \cap Z(R). \tag{2.3.4}$$

Replacing h' by $(k')^2$ in (2.3.4), where $k' \in S(R) \cap Z(R)$, we get

$$0 = d(h') = d((k')^2) = d(k')k' + k'd(k') = 2d(k')k'.$$

Since $char(R) \neq 2$, we arrive at

$$d(k^{'})k^{'} = 0 \text{ for all } k^{'} \in S(R) \cap Z(R).$$

Now since the centre of a prime ring is free from zero divisors, we get for each $k^{'} \in S(R) \cap Z(R)$ either $d(k^{'}) = 0$ or $k^{'} = 0$. Since $k^{'} = 0$ implies $d(k^{'}) = 0$, we may write

$$d(k^{'}) = 0 \text{ for all } k^{'} \in S(R) \cap Z(R). \tag{2.3.5}$$

Let $x \in Z(R)$. Since $char(R) \neq 2$, every $x \in Z(R)$ can be represented as $2x = h + k$, where $h \in H(R) \cap Z(R)$ and $k \in S(R) \cap Z(R)$. This implies that $2d(x) = d(2x) = d(h+k) = d(h) + d(k) = 0$. Since $char(R) \neq 2$, we get

$$d(x) = 0 \text{ for all } x \in Z(R). \tag{2.3.6}$$

Replacing y by $k^{'}y$ in (2.3.2), where $k^{'} \in S(R) \cap Z(R)$ and using (2.3.6), we arrive at

$$k^{'}(-[d(x), d(y^*)] + [d(y), d(x^*)]) = 0 \text{ for all } k^{'} \in S(R) \cap Z(R) \text{ and } x, y \in R.$$

Using the primeness of R and the fact that $S(R) \cap Z(R) \neq (0)$, we get

$$-[d(x), d(y^*)] + [d(y), d(x^*)] = 0 \text{ for all } x, y \in R. \tag{2.3.7}$$

On comparing (2.3.2) and (2.3.7), we obtain $2[d(x), d(y^*)] = 0$. Replacing y by y^* and using the fact that $char(R) \neq 2$, we conclude that $[d(x), d(y)] = 0$ for all $x, y \in R$. Therefore in view of [76], we get R is commutative. Now we consider the case

$$[d(x), x^*] + [d(x), x] + [x^*, d(x^*)] + [x, d(x^*)] = 0 \text{ for all } x \in R.$$

Replacing x by $h + k$, where $h \in H(R)$ and $k \in S(R)$, we get $4[d(k), h)] = 0$. Since $char(R) \neq 2$, we obtain

$$[d(k), h] = 0 \text{ for all } h \in H(R) \text{ and } k \in S(R). \tag{2.3.8}$$

Replacing h by $k_0k^{'}$, where $k_0 \in S(R)$ and $k^{'} \in S(R) \cap Z(R)$, we arrive at $([d(k), k_0])k^{'} =$

0. Using the primeness of R and since $S(R) \cap Z(R) \neq (0)$, we get

$$[d(k), k_0] = 0 \text{ for all } k, k_0 \in S(R). \tag{2.3.9}$$

Now since $char(R) \neq 2$, every $x \in R$ can be represented as $2x = h+k$, where $h \in H(R)$, $k \in S(R)$, so in view of equations (2.3.8) and (2.3.9), we are forced to conclude that

$$[d(k), x] = 0 \text{ for all } k \in S(R) \text{ and } x \in R. \tag{2.3.10}$$

for all $k \in S(R)$ and $x \in R$. That is, $d(k) \in Z(R)$ for all $k \in S(R)$. First we assume that $d(S(R)) = (0)$. Then, we have $d(x - x^*) = 0$ for all $x \in R$. That is, $d(x) = d(x^*)$ for all $x \in R$. Now for $k \in S(R)$ and $x \in R$, we have $0 = d(kx+x^*k) = kd(x)+d(x^*)k = kd(x) + d(x)k$ for all $x \in R$. This further implies that $k^2d(x) = d(x)k^2$ for all $x \in R$. Thus, in view of Lemma 1.3.11, we conclude that $k^2 \in Z(R)$ for all $k \in Z(R)$. Since $S(R) \cap Z(R) \neq (0)$, let $0 \neq k_0 \in S(R) \cap Z(R)$ and let k be an arbitrary element of $S(R)$. Then $(k + k_0)^2 = k^2 + k_0^2 + 2kk_0 \in Z(R)$ and hence $2kk_0 \in Z(R)$. Since $char(R) \neq 2$, we get $kk_0 \in Z(R)$ for all $k \in S(R)$ and $k_0 \in S(R) \cap Z(R)$. This further implies that $k \in Z(R)$ for all $k \in S(R)$ and hence R is normal. Thus R is commutative in view of Lemma 2.2.1. Now suppose $d(S(R)) \neq (0)$. For $k_o \in S(R)$ with $d(k_o) \neq 0$ and $k \in [S(R), S(R)]$, we have $d(k_okk_o) \in Z(R)$. The last expression can be written as $d(k_o)kk_o + k_okd(k_o) \in Z(R)$, since $d([S(R), S(R)]) = 0$. Thus $d(k_o)(k_ok + kk_o) \in Z(R)$ and hence $k_ok + kk_o \in Z(R)$ for all $k \in [S(R), S(R)]$. This implies that $d(k_ok + kk_o) \in Z(R)$ and hence $2d(k_o)k \in Z(R)$. Since $char(R) \neq 2$ and R is prime, the above relation yields that $k \in Z(R)$. That is, $[S(R), S(R)] \subseteq Z(R)$. Suppose $[S(R), S(R)] \neq (0)$ and let $k, k_o \in S(R)$ such that $[k, k_o] \neq 0$. Since $kk_ok \in S(R)$, we have $[k, kk_ok] = k[k, k_o]k = k^2[k, k_o] \in Z(R)$. This implies that $k^2 \in Z(R)$ and hence $k \in Z(R)$ for all $k \in S(R)$ as proved earlier. Therefore R is commutative in view of Lemma 2.2.1. Now suppose $[S(R), S(R)] = (0)$. Since $\overline{S(R)^2}$ is both a Lie ideal and a commutative subring of R , by Lemma 1.3.9, $k^2 \in Z(R)$ for all $k \in S(R)$ and hence $k \in Z(R)$ for all $k \in S(R)$. Thus, R is normal and hence R is commutative by Lemma 2.2.1. This completes the proof of the theorem. □

If we replace commutator by anti-commutator in Theorem 2.3.1, the corresponding result also holds.

Theorem 2.3.2. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If d is a nonzero derivation of R such that $d(x) \circ d(x^*) = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$, then R is commutative.*

Proof. By the assumption, we have $d(x) \circ d(x^*) = 0$ for all $x \in R$. This can be further written as

$$d(x)d(x^*) + d(x^*)d(x) = 0 \text{ for all } x \in R. \tag{2.3.11}$$

Replacing x by $x + y$ in (2.3.11), we get

$$d(x)d(y^*) + d(y)d(x^*) + d(x^*)d(y) + d(y^*)d(x) = 0 \text{ for all } x, y \in R. \tag{2.3.12}$$

Taking $x = h'$, where $h' \in H(R) \cap Z(R)$ in (2.3.11), we have $2d(h')^2 = 0$ for all $h' \in H(R) \cap Z(R)$. Since $char(R) \neq 2$, using the primeness of R we obtain

$$d(h') = 0 \text{ for all } h' \in H(R) \cap Z(R). \tag{2.3.13}$$

Using the same technique as we have used after (2.3.4), we finally arrive at

$$d(x) = 0 \text{ for all } x \in Z(R). \tag{2.3.14}$$

Replacing y by $k'y$, where $k' \in S(R) \cap Z(R)$ in (2.3.12) and using (2.3.14), we get

$$k'(-d(x)d(y^*) + d(y)d(x^*) + d(x^*)d(y) - d(y^*)d(x)) = 0$$

for all $h' \in H(R) \cap Z(R)$ and $x, y \in R$. Using the primeness of R and since $S(R) \cap Z(R) \neq (0)$, we have

$$-d(x)d(y^*) + d(y)d(x^*) + d(x^*)d(y) - d(y^*)d(x)) = 0 \text{ for all } x, y \in R. \tag{2.3.15}$$

On comparing (2.3.12) and (2.3.15), we obtain $2(d(x)d(y^*) + d(y^*)d(x)) = 0$ for all $x, y \in R$. Since $char(R) \neq 2$ and on replacing y by y^*, we finally arrive at $d(x) \circ d(y) = 0$ for all $x, y \in R$. Hence, R is commutative in view of Lemma 1.3.1. □

In the year 1995, Bell and Daif [34] showed that if R is a prime ring admitting a nonzero derivation d such that $d([x,y]) = 0$ for all $x, y \in R$, then R is commutative. This result was extended for semiprime rings in [54] by Daif. Further, for semiprime rings, Andima and Pajoohesh [11] showed that an inner derivation satisfying the above mentioned condition on a nonzro ideal of R must be zero on that ideal. Moreover, for semiprime rings with identity, they generalized this result to inner derivations of powers of x and y in [11]. Recently, many authors (see for example [11, 14, 21, 22, 34, 35, 55, 59, 60, 100, 112]) have obtained commutativity of prime and semiprime rings satisfying certain differential identities. In this section, we study the above mentioned result and some other results in the setting of prime rings with involution. We start with the following result.

Theorem 2.3.3. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If d is a nonzero derivation of R such that $d([x,x^*]) = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$, then R is commutative.*

Proof. By the assumption, we have

$$d([x,x^*]) = 0 \text{ for all } x \in R. \tag{2.3.16}$$

A lineralization of (2.3.16) yields that

$$d([x,y^*] + [y,x^*]) = 0 \text{ for all } x, y \in R. \tag{2.3.17}$$

Replacing y by xx^* in (2.3.17), we get

$$\begin{aligned} 0 &= d([x,xx^*] + [xx^*,x^*]) \\ &= d(x[x,x^*] + [x,x^*]x^*) \\ &= d(x)[x,x^*] + [x,x^*]d(x^*) \text{ for all } x \in R. \end{aligned}$$

That is,

$$d(x)[x,x^*] + [x,x^*]d(x^*) = 0 \text{ for all } x \in R. \tag{2.3.18}$$

Replacing x by $h+k$, where $h \in H(R)$, $k \in S(R)$, we obtain

$$d(h)[h,k] + d(k)[h,k] + [h,k]d(h) - [h,k]d(k) = 0 \tag{2.3.19}$$

for all $h \in H(R)$ and $k \in S(R)$. Replacing k by $-k$ in (2.3.19), we obtain

$$-d(h)[h,k] + d(k)[h,k] - [h,k]d(h) + [h,k]d(k) = 0 \tag{2.3.20}$$

for all $h \in H(R)$ and $k \in S(R)$. Adding (2.3.19) and (2.3.20) and using the fact $char(R) \neq 2$, we get

$$d(k)[h,k] - [h,k]d(k) = 0 \text{ for all } h \in H(R) \text{ and } k \in S(R). \tag{2.3.21}$$

Replacing k by $h_o k'$ in (2.3.21), where $h_o \in H(R)$ and $k' \in S(R) \cap Z(R)$, we arrive at

$$(d(k)[h,h_o] - [h,h_o]d(k))k' = 0 \text{ for all } h, h_0 \in H(R), k \in S(R) \text{ and } k' \in S(R) \cap Z(R).$$

Since the centre of a prime ring is free from zero divisors and $S(R) \cap Z(R) \neq (0)$, we get

$$d(k)[h,h_o] - [h,h_o]d(k) = 0 \text{ for all } h, h_o \in H(R) \text{ and } k \in S(R). \tag{2.3.22}$$

Now since every $x \in R$ can be represented as $2x = h + k$, where $h \in H(R)$, $k \in S(R)$, in view of equations (2.3.21) and (2.3.22), we obtain

$$d(k)[h,x] - [h,x]d(k) = 0 \text{ for all } h \in H(R), k \in S(R) \text{ and } x \in R. \tag{2.3.23}$$

If $h \in Z(R)$. Then $k^2 \in Z(R)$ for all $k \in S(R)$. Since $S(R) \cap Z(R) \neq (0)$, let $0 \neq k_0 \in S(R) \cap Z(R)$ and let k be an arbitrary element of $S(R)$. Then $(k + k_0)^2 = k^2 + k_0^2 + 2kk_0 \in Z(R)$ and hence $2kk_0 \in Z(R)$. Since $char(R) \neq 2$, we get $kk_o \in Z(R)$ for all $k \in S(R)$ and $k_o \in S(R) \cap Z(R)$. This further implies that $k \in Z(R)$ for all $k \in S(R)$. Hence $2x = h + k \in Z(R)$ for all $x \in R$. Since $char(R) \neq 2$, we get $R \subseteq Z(R)$. That is, R is commutative. If $h \notin Z(R)$, then the map defined by $D(x) = [x,h]$ for all $x \in R$ is a nonzero derivation and by (2.3.23) $d(k)D(x) = D(x)d(k)$. Thus, in view of the Lemma 1.3.11, we conclude that $d(k) \in Z(R)$ for all $k \in S(R)$. Hence using the same technique as used in the proof of Theorem 2.3.1, we get the required result. This completes the proof of the theorem. $\square$

If we replace commutator by anti-commutator in Theorem 2.3.3, the corresponding result also holds.

Theorem 2.3.4. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If d is a nonzero derivation of R such that $d(x \circ x^*) = 0$ for all $x \in R$.and $S(R) \cap Z(R) \neq (0)$, then R is commutative.*

Proof. By the assumption, we have

$$d(x \circ x^*) = 0 \text{ for all } x \in R.$$

The above equation can be further written as

$$d(x)x^* + xd(x^*) + d(x^*)x + x^*d(x) = 0 \text{ for all } x \in R. \tag{2.3.24}$$

Replacing x by $h \in H(R) \cap Z(R)$ in (2.3.24), and using the fact that $char(R) \neq 2$, we obtain

$$d(h)h = 0 \text{ for all } h \in H(R) \cap Z(R).$$

Now since the centre of a prime ring is free from zero divisors, we get for each $h \in H(R) \cap Z(R)$ either $d(h) = 0$ or $h = 0$. Since $h = 0$ implies $d(h) = 0$, we may write

$$d(h) = 0 \text{ for all } h \in H(R) \cap Z(R). \tag{2.3.25}$$

Using the same techniques as used after (2.3.4), we finally arrive at

$$d(x) = 0 \text{ for all } x \in Z(R). \tag{2.3.26}$$

Linearizing (2.3.24), we obtain

$$\begin{aligned} & d(x)y^* + d(y)x^* + xd(y^*) + yd(x^*) + d(x^*)y \\ + \quad & d(y^*)x + x^*d(y) + y^*d(x) = 0 \text{ for all } x, y \in R. \end{aligned} \tag{2.3.27}$$

Replacing y by $y_o \in Z(R)$ in (2.3.27) and making use of (2.3.26), we have

$$d(x)y_o^* + y_od(x^*) + d(x^*)y_o + y_o^*d(x) = 0 \text{ for all } y_o \in Z(R) \text{ and } x \in R. \tag{2.3.28}$$

In particular taking $y_o = h_o \in H(R) \cap Z(R)$ in the above equation, we get $2(d(x)h_o +$

$d(x^*)h_o) = 0$. Since $char(R) \neq 2$, we obtain $d(x)h_o + d(x^*)h_o = 0$. This can be further written as

$$d(x + x^*)h_o = 0 \text{ for all } h_o \in H(R) \cap Z(R) \text{ and } x \in R. \tag{2.3.29}$$

Using the primeness of R we get either $d(x + x^*) = 0$ or $H(R) \cap Z(R) = (0)$. But $H(R) \cap Z(R) = (0)$ implies that $S(R) \cap Z(R) = (0)$, which gives a contradiction since we have assumed $S(R) \cap Z(R) \neq (0)$. Therefore we left with the case $d(x + x^*) = 0$ for all $x \in R$. Replacing x by $h + k$ in the above equation, we get $2d(h) = 0$. Since $char(R) \neq 2$, we obtain $d(h) = 0$ for all $h \in H(R)$. Further $d(x + x^*) = 0$ implies that $d(x) = -d(x^*)$ for all $x \in R$. Replacing x by xh, where $h \in H(R)$ in the last expression we get $d(x)h = -hd(x^*)$, since $d(h) = 0$. This further implies that $d(x)h = hd(x)$ for all $x \in R$. Therefore in view of the Lemma 1.3.11, we conclude that $h \in Z(R)$ for all $h \in H(R)$. Hence R is commutative in view of Lemma 2.2.1. Thereby completing the proof of the theorem. $\square$

In [55], Daif and Bell established that a semiprime ring R admitting a derivation d such that $d([x, y]) \pm [x, y] = 0$ for all $x, y \in R$ must be commutative. Further, Argaç [14] generalized this result for a nonzero ideal of a semiprime ring. We prove the following theorem in the setting of rings with involution.

Theorem 2.3.5. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If d is a nonzero derivation of R such that $d([x, x^*]) \pm [x, x^*] = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$, then R is commutative.*

Proof. We first consider the case

$$d([x, x^*]) - [x, x^*] = 0 \text{ for all } x \in R. \tag{2.3.30}$$

A lineralization of (2.3.30) yields that

$$d([x, y^*] + [y, x^*]) - ([x, y^*] + [y, x^*]) = 0 \text{ for all } x, y \in R. \tag{2.3.31}$$

Replacing y by xx^* in (2.3.31), we get

$$\begin{aligned}
0 &= d(x[x,x^*]) + [x,x^*]x^*) - x[x,x^*] - [x,x^*]x^* \\
&= d(x)[x,x^*] + xd([x,x^*]) + d([x,x^*])x^* \\
&+ [x,x^*]d(x^*) - x[x,x^*] - [x,x^*]x^* \\
&= d(x)[x,x^*] + [x,x^*]d(x^*) \text{ for all } x \in R.
\end{aligned}$$

That is,

$$d(x)[x,x^*] + [x,x^*]d(x^*) = 0 \text{ for all } x \in R. \tag{2.3.32}$$

Hence using the same technique as used after (2.3.18) in the proof of Theorem 2.3.3, we get the required result.

By the same arguments, we obtain the same conclusion in case $d([x,x^*]) + [x,x^*] = 0$ for all $x \in R$. This proves the theorem. □

If we replace commutator by anti-commutator in Theorem 2.3.5, the corresponding result also holds.

Theorem 2.3.6. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If d is a nonzero derivation of R such that $d(x \circ x^*) \pm (x \circ x^*) = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$, then R is commutative.*

Proof. First we consider the case $d(x \circ x^*) - (x \circ x^*) = 0$ for all $x \in R$. This can be further written as

$$d(x)x^* + xd(x^*) + d(x^*)x + x^*d(x) - xx^* - x^*x = 0 \text{ for all } x \in R. \tag{2.3.33}$$

Linearization of (2.3.33) yields that

$$\begin{aligned}
&d(x)y^* + xd(y^*) + d(y^*)x + y^*d(x) + d(y)x^* + yd(x^*) \qquad (2.3.34)\\
+ \ &d(x^*)y + x^*d(y) - xy^* - y^*x - yx^* - x^*y = 0 \text{ for all } x, y \in R.
\end{aligned}$$

Replacing y by $h'x$ in (2.3.34), where $h' \in H(R) \cap Z(R)$ and making use of (2.3.33), we get $2(x \circ x^*)d(h') = 0$. Since $char(R) \neq 2$, we obtain

$$(x \circ x^*)d(h') = 0 \text{ for all } x \in R \text{ and } h' \in H(R) \cap Z(R). \tag{2.3.35}$$

Since the centre of a prime ring is free from zero divisors we get either $d(h') = 0$ for all $h' \in H(R) \cap Z(R)$ or $x \circ x^* = 0$ for all $x \in R$. Suppose

$$d(h') = 0 \quad \text{for all } h' \in H(R) \cap Z(R). \tag{2.3.36}$$

Using the same technique as used after (2.3.4), we finally arrive at

$$d(x) = 0 \text{ for all } x \in Z(R). \tag{2.3.37}$$

Replacing y by $y_o \in Z(R)$ in (2.3.34) and using (2.3.38), we arrive at

$$d(x)y_o^* + y_o^*d(x) + y_od(x^*) + d(x^*)y_o - xy_o^* - y_o^*x - y_ox^* - x^*y_o = 0 \tag{2.3.38}$$

for all $x \in R$ and $y_o \in Z(R)$. In particular, taking $y_o = h_o \in H(R) \cap Z(R)$ in (2.3.38), we obtain

$$(d(x + x^*) - (x + x^*))h_o = 0 \text{ for all } x \in R \text{ and } h_o \in H(R) \cap Z(R).$$

Using the primeness of R we get either $d(x + x^*) - (x + x^*) = 0$ for all $x \in R$ or $H(R) \cap Z(R) = (0)$. But $H(R) \cap Z(R) = (0)$ implies that $S(R) \cap Z(R) = (0)$, which gives a contradiction since we have assumed $S(R) \cap Z(R) \neq (0)$. Therefore we are left with the case

$$d(x + x^*) - (x + x^*) = 0 \text{ for all } x \in R. \tag{2.3.39}$$

Replacing x by $h + k$ in (2.3.39) where $h \in H(R)$, $k \in S(R)$, we obtain

$$d(h) = h \text{ for all } h \in H(R). \tag{2.3.40}$$

Taking $y_o = k_o \in S(R) \cap Z(R)$ in (2.3.38), we obtain

$$(d(x - x^*) - (x - x^*))k_o = 0 \text{ for all } x \in R \text{ and } k_o \in S(R) \cap Z(R).$$

Again using the primeness of R and the fact that $S(R) \cap Z(R) \neq (0)$, we get

$$d(x - x^*) - (x - x^*) = 0 \text{ for all } x \in R. \tag{2.3.41}$$

Replacing x by $h + k$ in (2.3.41), where $h \in H(R)$ and $k \in S(R)$, we obtain

$$d(k) = k \text{ for all } k \in S(R). \tag{2.3.42}$$

Since every $x \in R$ can be represented as $2x = h + k$, $h \in H(R)$, $k \in S(R)$, it follows from (2.3.40) and (2.3.42) that $2d(x) = d(2x) = d(h + k) = d(h) + d(k) = h + k = 2x$. Since $char(R) \neq 2$, we obtain $d(x) = x$ for all $x \in R$. Therefore in view of the relation (2.3.37), we get $Z(R) = (0)$. Which gives a contradiction. Therefore we are left with the case $x \circ x^* = 0$ for all $x \in R$. Linearization of the last relation yields that $xoy^* + yox^* = 0$ for all $x, y \in R$. Replacing y by x^2 and using the fact that $x \circ x^* = 0$, we obtain

$$\begin{aligned}
0 &= x \circ (x^*)^2 + x^* \circ x^2 \\
&= (x \circ x^*)x^* - x^*[x, x^*] + (x^* \circ x)x - x[x^*, x] \\
&= x[x, x^*] - x^*[x, x^*] \\
&= (x - x^*)[x, x^*] \text{ for all } x \in R.
\end{aligned}$$

Substituting $h + k$ for x, where $h \in H(R)$, $k \in S(R)$, we get $2k([k, h] - [h, k]) = 0$ and hence $4k[h, k] = 0$ for all $h \in H(R)$ and $k \in S(R)$. Since $char(R) \neq 2$, the above relation forces that $k[h, k] = 0$ for all $h \in H(R)$ and $k \in S(R)$. Replacing k by $k + k_1$ where $k_1 \in S(R) \cap Z(R)$, we obtain $k_1[h, k] = 0$ for all $h \in H(R)$, $k \in S(R)$ and $k_1 \in S(R) \cap Z(R)$. Using the primeness of R and the fact that $S(R) \cap Z(R) \neq (0)$, we conclude that $[h, k] = 0$ for all $h \in H(R)$ and $k \in S(R)$. That is, R is normal. Hence, application of Lemma 2.2.1 yields the required result. That is, R is commutative.

By the same arguments, we obtain the same conclusion in case $d(x \circ x^*) + (x \circ x^*) = 0$ for all $x \in R$. This proves the theorem. $\square$

Theorem 2.3.7. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If d is a nonzero derivation of R such that $d([x, x^*]) \pm (x \circ x^*) = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$, then R is commutative.*

Proof. By the given hypothesis, we have

$$d([x,x^*]) - (x \circ x^*) = 0 \text{ for all } x \in R. \tag{2.3.43}$$

Linearizing (2.3.43), we get

$$d([x,y^*]) - (x \circ y^*) + d([y,x^*]) - (y \circ x^*) = 0 \text{ for all } x, y \in R. \tag{2.3.44}$$

Replacing y by xx^* in (2.3.44) and making use of (2.3.43), we obtain

$$d(x)[x,x^*] + [x,x^*]d(x^*) + xd([x,x^*]) - 2x^2x^* + x[x,x^*] = 0 \text{ for all } x \in R.$$

This can be further written as $d(x)[x,x^*] + [x,x^*]d(x^*) + xd([x,x^*]) - x(x \circ x^*) = 0$. Using (2.3.43) again we finally get

$$d(x)[x,x^*] + [x,x^*]d(x^*) = 0 \text{ for all } x \in R. \tag{2.3.45}$$

The last expression is same as the equation (2.3.18) and hence, by using similar approach as we have used after (2.3.18) in the proof of Theorem 2.3.3, we get the required result.

By the same arguments, we obtain the same conclusion in case $d[x,x^*]+(x\circ x^*) = 0$ for all $x \in R$. This proves the theorem. □

Using the similar approach as in Theorem 2.3.7, we have the following result.

Theorem 2.3.8. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If d is a nonzero derivation of R such that $d(x \circ x^*) \pm [x,x^*] = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$, then R is commutative.*

Theorem 2.3.9. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If d is a nonzero derivation of R such that $d(xx^*) \pm xx^* = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$, then R is commutative.*

Proof. By the given hypothesis, we have

$$d(xx^*) \pm xx^* = 0 \text{ for all } x \in R. \tag{2.3.46}$$

Replacing x by x^* in (2.3.46), we get

$$d(x^*x) \pm x^*x = 0 \text{ for all } x \in R. \tag{2.3.47}$$

On comparing equations (2.3.46) and (2.3.47), we arrive at

$$d([x, x^*]) \pm [x, x^*] = 0 \text{ for all } x \in R. \tag{2.3.48}$$

Hence the result follows in view of Theorem 2.3.5, which completes the proof of the theorem. □

Theorem 2.3.10. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If d is a nonzero derivation of R such that $d(x)d(x^*) \pm xx^* = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$, then R is commutative.*

Proof. First we consider the case

$$d(x)d(x^*) - xx^* = 0 \text{ for all } x \in R. \tag{2.3.49}$$

Linearization of (2.3.49) yields that

$$d(x)d(y^*) + d(y)d(x^*) - xy^* - yx^* = 0 \text{ for all } x, y \in R. \tag{2.3.50}$$

Replacing y by hx in (2.3.50), where $h \in H(R) \cap Z(R)$ and making use of (2.3.49), we get

$$d(x)x^*d(h) + d(h)xd(x^*) = 0 \text{ for all } h \in H(R) \cap Z(R) \text{ and } x \in R. \tag{2.3.51}$$

This can be further written as $d(h)d(xx^*) = 0$ for all $h \in H(R) \cap Z(R)$ and $x \in R$. Using the primeness of R, we have either $d(h) = 0$ for all $h \in H(R) \cap Z(R)$ or $d(xx^*) = 0$ for all $x \in R$. Suppose $d(h) = 0$ for all $h \in H(R) \cap Z(R)$. Then using the same argument as used after (2.3.4), we finally arrive at

$$d(x) = 0 \text{ for all } x \in Z(R). \tag{2.3.52}$$

Replacing y by $h_o \in H(R) \cap Z(R)$ in (2.3.50) and making use of (2.3.52), we have

$xh_o + h_o x^* = 0$. This can be further written as $(x + x^*)h_o = 0$ for all $h_o \in H(R) \cap Z(R)$ and $x \in R$. Using the primeness of R and the fact that $S(R) \cap Z(R) \neq (0)$, we get $x + x^* = 0$ for all $x \in R$. Replacing x by $h + k$, where $h \in H(R)$, $k \in S(R)$ in the above relation, we obtain $2h = 0$ for all $h \in H(R)$. Since $char(R) \neq 2$, we get $H(R) = (0)$. Hence R is commutative in view of Lemma 2.2.1. Therefore we are left with the case $d(xx^*) = 0$ for all $x \in R$. This further implies that

$$d(x)x^* + xd(x^*) = 0 \text{ for all } x \in R. \tag{2.3.53}$$

Replacing x by $h \in H(R) \cap Z(R)$, we $2d(h)h = 0$. Since $char(R) \neq 2$, we arrive at $d(h)h = 0$ for all $h \in H(R) \cap Z(R)$. Now since the centre of a prime ring is free from zero divisors, we get for each $h \in H(R) \cap Z(R)$ either $d(h) = 0$ or $h = 0$. Since $h = 0$ also implies $d(h) = 0$, we may write $d(h) = 0$ for all $h \in H(R) \cap Z(R)$. Again using the same argument as we have used in the proof of Theorem 2.3.1, we get $d(x) = 0$ for all $x \in Z(R)$. Linearizing (2.3.53) and using the same argument as we have used after (2.3.52), we arrive at $hd(x + x^*) = 0$ for all $h \in H(R)$ and $x \in R$. Thus by the primeness of R, we obtain either $H(R) \cap Z(R) = (0)$ or $d(x + x^*) = 0$ for all $x \in R$. Since $S(R) \cap Z(R) \neq (0)$, we get $d(x + x^*) = 0$ for all $x \in R$. Using the same argument as in the proof of the Theorem 2.3.4, we get $h \in Z(R)$ for all $h \in H(R)$. Hence R is commutative in view of Lemma 2.2.1 . Thereby completing the proof of the theorem.

By the same arguments, we obtain the same conclusion in case $d(x)d(x^*) + xx^* = 0$ for all $x \in R$. This completes the proof. □

We conclude this chapter with the following example which shows that the condition $S(R) \cap Z(R) \neq (0)$ in the hypothesis of Theorem 2.3.1, Theorem 2.3.3, Theorem 2.3.4 and Theorem 2.3.5 is not superfluous.

Example 2.3.1. Let

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c, d \in Z \right\}.$$

Of course R with matrix addition and matrix multiplication is a prime ring. Define mappings $d : R \longrightarrow R$, and $* : R \longrightarrow R$ such that

$$d\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix} \qquad a, b, c, d \in Z$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad a, b, c, d \in Z.$$

Then it is easy to verify that $x^* = x$ for all $x \in Z(R)$, and hence $Z(R) \subseteq H(R)$. Which intern shows that $S(R) \cap Z(R) = (0)$. Moreover, d is a nonzero derivation and the following conditions: (i) $[d(x), d(x^*)] = 0$, (ii) $d([x, x^*]) = 0$, (iii) $d(x \circ x^*) = 0$, (iv) $d([x, x^*]) \pm [x, x^*] = 0$ for all $x \in R$, are satisfied. However, R is not commutative. Hence, the condition $S(R) \cap Z(R) \neq (0)$ is crucial in Theorem 2.3.1, Theorem 2.3.3, Theorem 2.3.4 and Theorem 2.3.5.

CHAPTER-3

On Left Centralizers of Prime Rings with Involution

Chapter 3

On Left Centralizers of Prime Rings with Involution

3.1 Introduction

A left (resp. right) centralizer of R is an additive mapping $T : R \to R$ which satisfies $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) for all $x, y \in R$. If $a \in R$, then the mapping $x \mapsto ax$ is a left centralizer and $x \mapsto xa$ is a right centralizer on R. An additive mapping $T : R \to R$ is called a two sided centralizer in case T is a left and a right centralizer of R. Historically, the study of such mappings was initiated by Zalar [146] and subsequently studied by Vukman (viz.; [131, 132, 133, 137] and the references there in). This concept appears naturally in C^*-algebras. In ring theory, it is more common to work with module homomorphisms. Ring theorists would write that $T : R_R \to R_R$ is a homomorphism of a ring module R into itself. For a semiprime ring R all such homomorphisms are of the form $T(x) = qx$ for all $x \in R$, where q is an element of Martindale right ring of quotients $Q_r(R)$ (see [37, Chapter 2]). Starting with the paper by Zalar [146], during the last some decades the study of centralizers became an active area of research in semi(prime) rings, C^*-algebras and H^*-algebras (see for instance, [2, 5, 13, 15, 62, 63, 64, 106, 109, 111, 121, 138, 139, 141, 144] for details.) In the year 1992, Daif and Bell [55] established that a semiprime ring admitting a derivation d such that $d([x, y]) \pm [x, y] = 0$ for all $x, y \in R$ must be commutative. In [14], Argaç generalized this result for a nonzero ideal of semiprime rings. Recently, Ashraf and Ali [15] studied same result in the setting of prime rings involving left centralizers.

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In fact, they proved the following result: If a prime ring R admits a nonzero left centralizer(multiplier) $T : R \to R$ such that $T([x, y]) \pm [x, y] = 0$ with $T(x) \neq x$ for all $x, y \in I$, a nonzero ideal of R, then R is commutative. In the same paper some related results involving left centralizers have also been discussed. In [109], Oukhtite established similar problems for certain situations involving left centralizers acting on Lie ideals in rings with involution.

Section 3.2 deals with the study of normality of prime rings with involution involving left centralizers. In fact, we prove that if a prime ring with involution $*$ of characteristic different from two admits a nonzero left centralizer T such that $[T(x), x^*] = 0$ for all $x \in R$, then R is normal. Further, we characterize normal rings and two sided centralizers among all prime rings with involution satisfying certain identities involving left centralizers.

In Section 3.3, we study the result of Bell and Daif [34, Theorem 3] in the setting of rings with involution, and consequently it is shown that a prime ring with involution $*$ of characteristic different from two admits a nonzero left centralizer $T : R \to R$ such that $T([x, x^*]) = 0$ for all $x \in R$ with $S(R) \cap Z(R) \neq (0)$, must be commutative.

The last section is devoted to the study of commutativity of prime ring with involution. Besides proving some other results, we study a result due to Ashraf and Ali [15, Theorem 2.1] for prime rings with involution, and prove that a prime ring with involution $*$ of characteristic different from two admits a nonzero left centralizer $T : R \to R$ such that $T([x, x^*]) - [x, x^*] = 0$ for all $x \in R$ with $S(R) \cap Z(R) \neq (0)$, must be commutative.

We shall restrict our attention on left centralizers since all results presented in this chapter are also true for right centralizers because of left and right symmetry.

3.2 The condition $[T(x), x^*] = 0$

In the year 1957, Posner [115] proved that if R is a prime ring and d is a nonzero derivation such that $[d(x), x] = 0$ for all $x \in R$, then R is commutative. In Chapter 2, we establish a $*$-version of the above mentioned result as follows: Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If d is a nonzero derivation of R such that $[d(x), x^*] = 0$ for all $x \in R$ and $d(Z(R)) \neq (0)$, then R is normal. In the present section, we continue this study and obtain a similar result for left centralizers. Moreover, we also characterize normal rings and two sided centralizers among all prime rings with involution satisfying certain identities involving left centralizers.

We begin our discussion with the following lemma:

Lemma 3.2.1. *Let R be a prime ring with involution $*$. If T is a nonzero left centralizer of R such that $T(xx^*) = 0$ for all $x \in R$ or $T(x^*x) = 0$ for all $x \in R$, then R is normal.*

Proof. In view of our hypothesis, we have $T(xx^*) = 0$ for all $x \in R$. This can be further written as $T(x)x^* = 0$ for all $x \in R$. Linearizing the last relation, we obtain $T(x)y^* + T(y)x^* = 0$ for all $x, y \in R$. Substituting yx for y in the above expression and using the given hypothesis we find that $T(y)xx^* = 0$ for all $x, y \in R$. Replacing x by x^* in the last relation, we get $T(y)x^*x = 0$ for all $x, y \in R$. Last two relations yield that $T(y)[x, x^*] = 0$ for all $x, y \in R$. Replace y by yr to get $T(y)r[x, x^*] = 0$ for all $x, y, r \in R$ i.e., $T(y)R[x, x^*] = (0)$ for all $x,\ y \in R$. Thus, by the primeness of R we conclude that either $T(y) = 0$ for all $y \in R$ or $[x, x^*] = 0$ for all $x \in R$. Since $T(y) = 0$ for all $y \in R$, gives a contradiction. Thus the only possibility is $[x, x^*] = 0$ for all $x \in R$. Which proves that R is normal. □

By the similar arguments, we obtain the same conclusion in the case $T(x^*x) = 0$ for all $x \in R$. This proves the lemma.

Lemma 3.2.2. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. Suppose there exists $a \notin Z(R)$ such that $[h, a] = 0$ for all $h \in H(R)$, then R is normal.*

Proof. We have

$$[h, a] = 0 \text{ for all } h \in H(R). \tag{3.2.1}$$

If $h \in H(R)$, $k \in S(R)$, then $hk - kh \in H(R)$ and hence from (3.2.1), we obtain $[hk, a] - [kh, a] = 0$ for all $h \in H(R)$ and $k \in S(R)$. This implies that

$$h[k, a] + [h, a]k - k[h, a] - [k, a]h = 0 \text{ for all } h \in H(R) \text{ and } k \in S(R).$$

Application of (3.2.1) yields that

$$h[k, a] - [k, a]h = 0 \text{ for all } h \in H(R) \text{ and } k \in S(R). \tag{3.2.2}$$

Also we have

$$h[h_1, a] - [h_1, a]h = 0 \text{ for all } h, h_1 \in H(R). \tag{3.2.3}$$

Since $char(R) \neq 2$, every $x \in R$ can be represented as $2x = h_1 + k$ where $h_1 \in H(R)$, $k \in S(R)$ and therefore making use of (3.2.2) and (3.2.3), we obtain

$$2h[x, a] - 2[x, a]h = 0 \text{ for all } x \in R \text{ and } h \in H(R).$$

Since $char(R) \neq 2$, the last relation forces that

$$h[x, a] - [x, a]h = 0 \text{ for all } x \in R \text{ and } h \in H(R). \tag{3.2.4}$$

Now, since the mapping $x \mapsto [x, a]$ is a nonzero derivation and so in view of Lemma 1.3.11, we conclude that $h \in Z(R)$ for all $h \in H(R)$. Thereby proving R is normal. $\square$

We shall start our investigations with our first theorem which is inspired by the work of Divinsky [58].

Theorem 3.2.1. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If T is a nonzero left centralizer of R such that $[T(x), x^*] = 0$ for all $x \in R$, then R is normal.*

Proof. By the assumption, we have $[T(x), x^*] = 0$ for all $x \in R$. Replacing x by x^*, we obtain $[T(x^*), x] = 0$ for all $x \in R$. Let $f : R \to R$ be defined by $f(x) = T(x^*)$ for all $x \in R$. Then, it is easy to verify that f is a reverse left $*$-centralizer and hence $[f(x), x] = 0$ for all $x \in R$. In view of Lemma 1.3.5, we conclude that $f(x) = \mu x + \nu(x)$ for all $x \in R$, where $\mu \in C$, the extended centroid of R and $\nu : R \to C$ is an additive mapping. Define a new map $g : R \to R$ such that $g(x) = f(x)^*$ for all $x \in R$. Then clearly g is a left R-module homomorphism (i.e., right centralizer). Hence there exists $p \in Q_r(R)$ such that $g(x) = xp$ for all $x \in R$ (see [37, Chapter 2] for details). Therefore, we obtain $f(x) = \lambda x^*$ for all $x \in R$, where $\lambda = p^*$. Hence, we get

$$\lambda x^* - \mu x \in C \text{ for all } x \in R.$$

Since the identity involves involution, so it is a functional identity or the so called g-

identity (see [37, Chapter 6]). In view of Lemma 1.3.2, we conclude that $\lambda x^* - \mu x \in C$ for all $x \in Q_s(R)$, the symmetric ring of quotients. Note that $Q_s(R)$ has the identity element 1. Replacing x by 1 in the above expression, we see that $\lambda - \mu \in C$. This implies that $[\lambda, y] = 0$ for all $y \in Q_s(R)$. Thus,

$$T(x) = f(x^*) = \lambda x \text{ for all } x \in R,$$

where $\lambda \in C$. Since $T \neq 0$, it follows that $\lambda \neq 0$. Thus, we conclude that $0 = [T(x), x^*] = [\lambda x, x^*] = \lambda[x, x^*]$ for all $x \in R$. Hence, by the primeness of R, R is normal. This proves the theorem completely. $\square$

The above theorem has following interesting consequence:

Corollary 3.2.1. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If T is a nonzero left centralizer of R such that $[T(x), x^*] = 0$ for all $x \in R$, then there exists $\lambda \in C$, the extended centroid of R such that $T(x) = \lambda x$ for all $x \in R$.*

Theorem 3.2.2. *Let R be a noncommutative prime ring with involution $*$ such that $char(R) \neq 2$. If T_1 and T_2 are two nonzero left centralizers of R such that $T_1(x)x^* - x^*T_2(x) = 0$ for all $x \in R$, then R is normal.*

Proof. By the given hypothesis, we have

$$T_1(x)x^* - x^*T_2(x) = 0 \text{ for all } x \in R. \tag{3.2.5}$$

On linearizing (3.2.5), we get

$$T_1(x)y^* + T_1(y)x^* - x^*T_2(y) - y^*T_2(x) = 0 \text{ for all } x, y \in R. \tag{3.2.6}$$

Replacing y by xy in (3.2.6), we arrive at

$$T_1(x)y^*x^* + T_1(x)yx^* - x^*T_2(x)y - y^*x^*T_2(x) = 0 \text{ for all } x, y \in R. \tag{3.2.7}$$

Using (3.2.5) in (3.2.7), we obtain $T_1(x)y^*x^* + T_1(x)yx^* - T_1(x)x^*y - y^*T_1(x)x^* = 0$. This can be further written as $[T_1(x), y^*]x^* + T_1(x)[y, x^*] = 0$ for all $x, y \in R$. Since $T_1(x) = \lambda_1 x$ (where $\lambda_1 \in Q_r(R)$) for all $x \in R$. Thus $[\lambda_1 x, y^*]x^* + \lambda_1 x[y, x^*] = 0$ for all $x, y \in R$. Since the above identity is a g-identity (see [37, Chapter 6]). In view of

Lemma 1.3.2, we conclude that $[\lambda_1 x, y^*]x^* + \lambda_1 x[y, x^*] = 0$ for all $x, y \in Q_s(R)$, the symmetric ring of quotients. Note that $Q_s(R)$ has the identity element 1. Replacing x by 1 in the above expression, we see that $[\lambda_1, y] = 0$ for all $y \in Q_s(R)$. Thus,

$$T_1(x) = \lambda_1 x \text{ for all } x \in R,$$

where $\lambda_1 \in C$. Since $T_1 \neq 0$, it follows that $\lambda_1 \neq 0$. Also $T_2(x) = \lambda_2 x$, where $\lambda_2 \in Q_l(R)$. Hence from (3.2.5), $\lambda_1 x x^* - x^* \lambda_2 x = 0$ for all $x \in R$. Since the above identity is a g-identity. Thus by Lemma 1.3.2, we obtain $\lambda_1 x x^* - x^* \lambda_2 x = 0$ for all $x \in Q_s(R)$, the symmetric ring of quotients. Replacing x by 1 in the above expression, we see that $\lambda_1 = \lambda_2 = \lambda$ (say). Thus $T_2(x) = T_1(x) = \lambda x$ for all $x \in R$, where $0 \neq \lambda \in C$. Hence, we conclude that $0 = T_1(x)x^* - x^* T_2(x) = \lambda x x^* - x^* \lambda x = \lambda(x x^* - x^* x)$ for all $x \in R$. Thus by the primeness of R, R is normal. This proves the theorem completely. $\square$

As an immediate consequence of Theorem 3.2.2, we have the following corollary:

Corollary 3.2.2. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If T_1 and T_2 are two nonzero left centralizers of R such that $T_1(x)x^* - x^* T_2(x) = 0$ for all $x \in R$, then there exists $\lambda \in C$, the extended centroid of R such that $T_1(x) = T_2(x) = \lambda x$ for all $x \in R$.*

It would be interesting to know whether Theorem 3.2.1 and Theorem 3.2.2 hold in the case of arbitrary rings. Following examples justify this fact:

Example 3.2.1. *Let F be a field and* $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \,\middle|\, a, b, c \in F \right\}$. *Define mappings $T : R \longrightarrow R$, and $* : R \longrightarrow R$ such that*

$$T\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & c & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, it is easy to verify that T satisfies all the requirements of Theorem 3.2.1. However, R is not normal.

Example 3.2.2. *Consider the ring as in Example 3.2.1, and define mappings $T_1, T_2 :$ $R \longrightarrow R$ such that* $T_1\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, T_2\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$

Then, it is straightforward to check that T_1 and T_2 satisfy all the requirements of Theorem 3.2.2. However, R is not normal.

The aim of the rest of this section is to characterize both sided centralizers and normal rings among all prime rings with involution involving certain identities. We begin with the following result:

Theorem 3.2.3. *Let R be a prime ring with involution $*$. If T is a left centralizer of R such that $T(xx^*) \pm xx^* = 0$ for all $x \in R$, then either T is a centralizer or R is normal.*

Proof. First we consider the case $T(xx^*) - xx^* = 0$ for all $x \in R$. This can be further written as

$$T(x)x^* - xx^* = 0 \text{ for all } x \in R. \tag{3.2.8}$$

Linearizing the above relation, we get

$$T(x)y^* + T(y)x^* - xy^* - yx^* = 0 \text{ for all } x, y \in R. \tag{3.2.9}$$

Replacing y by yx in (3.2.9), we obtain

$$T(x)x^*y^* + T(y)xx^* - xx^*y^* - yxx^* = 0 \text{ for all } x, y \in R.$$

Application of (3.2.8) yields that

$$T(y)xx^* - yxx^* = 0 \text{ for all } x, y \in R. \tag{3.2.10}$$

Substituting zy for y in (3.2.10), we have

$$T(z)yxx^* - zyxx^* = 0 \text{ for all } x, y, z \in R. \tag{3.2.11}$$

Left multiplication to (3.2.10) by z yields that

$$zT(y)xx^* - zyxx^* = 0 \text{ for all } x, y, z \in R. \tag{3.2.12}$$

Subtracting (3.2.11) from (3.2.12), we obtain

$$zT(y)xx^* - T(z)yxx^* = 0 \text{ for all } x, y, z \in R. \tag{3.2.13}$$

Substituting yr for y in (3.2.13) to get $zT(y)rxx^* - T(z)yrxx^* = 0$ for all $x, y, z, r \in R$. Which can be further written as

$$(zT(y) - T(z)y)rxx^* = 0 \text{ for all } x, y, z, r \in R. \tag{3.2.14}$$

Replacing x by x^* in (3.2.14), we find that

$$(zT(y) - T(z)y)rx^*x = 0 \text{ for all } x, y, z, r \in R. \tag{3.2.15}$$

Subtracting (3.2.15) from (3.2.14), we obtain

$$(zT(y) - T(z)y)r[x, x^*] = 0 \text{ for all } x, y, z, r \in R.$$

This implies that $(zT(y) - T(z)y)R[x, x^*] = (0)$ for all $x, y, z \in R$. Thus by the primeness of R, we have either $zT(y) - T(z)y = 0$ for all $y, z \in R$ or $[x, x^*] = 0$ for all $x \in R$. Now if $zT(y) - T(z)y = 0$ for all $y, z \in R$. That is, $zT(y) = T(z)y$ for all $y, z \in R$. Then T is also a right centralizer of R and hence a centralizer of R. On the other hand if $[x, x^*] = 0$ for all $x \in R$, then R is normal.

By the same arguments, we obtain the same conclusion in case $T(xx^*) + xx^* = 0$ for all $x \in R$. This proves the theorem. □

By similar arguments as above with necessary variation, we can prove the following theorem:

Theorem 3.2.4. *Let R be a prime ring with involution $*$. If T is a left centralizer of R such that $T(x^*x) \pm x^*x = 0$ for all $x \in R$, then either T is a centralizer or R is normal.*

Theorem 3.2.5. *Let R be a prime ring with involution $*$. If T is a left centralizer of R such that $xT(x^*) \pm T(x)x^* = 0$ for all $x \in R$, then either T is a centralizer or R is normal.*

Proof. First we consider the case

$$xT(x^*) - T(x)x^* = 0 \text{ for all } x \in R. \tag{3.2.16}$$

Linearizing the above relation, we get

$$xT(y^*) - T(x)y^* + yT(x^*) - T(y)x^* = 0 \text{ for all } x, y \in R. \tag{3.2.17}$$

Replacing y by yx in (3.2.17) and using (3.2.16), we obtain

$$T(y)xx^* - yxT(x^*) = 0 \text{ for all } x, y \in R. \tag{3.2.18}$$

Substituting zy for y in (3.2.18), we have

$$T(z)yxx^* - zyxT(x^*) = 0 \text{ for all } x, y, z \in R. \tag{3.2.19}$$

Left multiplying (3.2.18) by z yields that

$$zT(y)xx^* - zyxT(x^*) = 0 \text{ for all } x, y, z \in R. \tag{3.2.20}$$

Subtracting (3.2.19) from (3.2.20), we obtain

$$T(z)yxx^* - zT(y)xx^* = 0 \text{ for all } x, y, z \in R. \tag{3.2.21}$$

Substituting yr for y in (3.2.21) we get

$$(zT(y) - T(z)y)rxx^* = 0 \text{ for all } x, y, z, r \in R. \tag{3.2.22}$$

The above equation is same as (3.2.14) and henceforward using the same approach as we have used in the last paragraph of the proof of Theorem 3.2.3, we get the required result. This proves the theorem. □

Theorem 3.2.6. *Let R be a prime ring with involution $*$. If T is a left centralizer of R such that $T(x)T(x^*) \pm xx^* = 0$ for all $x \in R$, then either T is a centralizer or R is normal.*

Proof. First we consider the situation

$$T(x)T(x^*) - xx^* = 0 \text{ for all } x \in R. \tag{3.2.23}$$

Replacing x by $x + y$, we get

$$T(x)T(y^*) - xy^* + T(y)T(x^*) - yx^* = 0 \text{ for all } x, y \in R. \tag{3.2.24}$$

Substituting yx for y in (3.2.24), we obtain

$$T(x)T(x^*)y^* - xx^*y^* + T(y)xT(x^*) - yxx^* = 0 \text{ for all } x, y \in R.$$

In view of (3.2.23), the above expression reduces to

$$T(y)xT(x^*) - yxx^* = 0 \text{ for all } x, y \in R. \tag{3.2.25}$$

Replace y by zy in (3.2.25) to get

$$T(z)yxT(x^*) - zyxx^* = 0 \text{ for all } x, y, z \in R. \tag{3.2.26}$$

Left multiplying (3.2.25) by z, we get

$$zT(y)xT(x^*) - zyxx^* = 0 \text{ for all } x, y, z \in R. \tag{3.2.27}$$

Subtracting (3.2.27) from (3.2.26), we obtain

$$zT(y)xT(x^*) - T(z)yxT(x^*) = 0 \text{ for all } x, y, z \in R. \tag{3.2.28}$$

Substituting yr for y in (3.2.28), we find that

$$zT(y)rxT(x^*) - T(z)yrxT(x^*) = 0 \text{ for all } x, y, z, r \in R.$$

This implies that

$$(zT(y) - T(z)y)rxT(x^*) = 0 \text{ for all } x, y, z, r \in R. \tag{3.2.29}$$

That is, $(zT(y) - T(z)y)RxT(x^*) = (0)$ for all $x, y, z \in R$. Thus by the primeness of R, we find that either $zT(y) - T(z)y = 0$ for all $y, z \in R$ or $xT(x^*) = 0$ for all $x \in R$. If $zT(y) - T(z)y = 0$ i.e., $T(z)y = zT(y)$ for all $y, z \in R$, then T is also a right centralizer and hence a centralizer on R. On the other hand, suppose $xT(x^*) = 0$ for all $x \in R$. This gives $T(y)xT(x^*) = 0$ for all $x, y \in R$. Hence (3.2.25) reduces to $yxx^* = 0$ for all $x, y \in R$. Replacing y by yr in the above relation, we obtain

$$yrxx^* = 0 \text{ for all } x, y, r \in R. \tag{3.2.30}$$

Replacing x by x^* in (3.2.30), we have

$$yrx^*x = 0 \text{ for all } x, y, r \in R. \tag{3.2.31}$$

Subtracting (3.2.31) from (3.2.30), we obtain

$$yr[x, x^*] = 0 \quad \text{for} \quad \text{all } x, y, r \in R.$$

This implies that $[x, x^*]R[x, x^*] = (0)$ for all $x \in R$. Since R is prime, the last expression forces that $[x, x^*] = 0$ for all $x \in R$.

Similar conclusion holds for the case $T(x)T(x^*) + xx^* = 0$ for all $x \in R$. This finishes the second case, and so the theorem is proved. □

Using similar approach with necessary variations we can establish the following:

Theorem 3.2.7. *Let R be a prime ring with involution $*$. If T is a left centralizer of R such that $T(x^*)T(x) \pm x^*x = 0$ for all $x \in R$, then either T is a centralizer or R is normal.*

3.3 The condition $T([x, x^*]) = 0$

In [34], Bell and Daif proved that if R is a prime ring admitting a nonzero derivation d such that $d([x, y]) = 0$ for all $x, y \in R$, then R is commutative. Further, this result was extended for semiprime ring in [54]. Recently, Andima and Pajoohesh [11] proved that an inner derivation satisfying the above mentioned condition on a nonzero ideal of R must be zero on that ideal. Motivated by the above mentioned results, we in Chapter 2 generalized the above mentioned result in the setting of prime rings with involution and prove the following result: Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. Let d be a derivation of R such that $d([x, x^*]) = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$. Then, R is commutative. The natural question here is whether an analogue holds true for left centralizers. Theorem 3.3.1 answers to this question in the affirmative.

We begin our investigation with the following proposition.

Proposition 3.3.1. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If $xox^* = 0$ for all $x \in R$ or $[x, x^*] = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$, then R is commutative.*

Proof. We have $xox^* = 0$ for all $x \in R$. Linearization of the above relation yields that $xoy^* + yox^* = 0$ for all $x, y \in R$. Replacing y by x^2 in the last relation and using the given hypothesis, we obtain

$$\begin{aligned}
0 &= xo(x^*)^2 + x^*ox^2 \\
&= (xox^*)x^* - x^*[x, x^*] + (x^*ox)x - x[x^*, x] \\
&= x[x, x^*] - x^*[x, x^*] \\
&= (x - x^*)[x, x^*] \text{ for all } x \in R.
\end{aligned}$$

Substituting $h + k$ for x, where $h \in H(R)$, $k \in S(R)$, we get $2k([k, h] - [h, k]) = 0$ and hence $4k[h, k] = 0$ for all $h \in H(R)$ and $k \in S(R)$. Since $char(R) \neq 2$, the above relation forces that $k[h, k] = 0$ for all $h \in H(R)$ and $k \in S(R)$. Replacing k by $k + k_1$ where $k_1 \in S(R) \cap Z(R)$, we obtain $k_1[h, k] = 0$ for all $h \in H(R)$, $k \in S(R)$ and $k_1 \in S(R) \cap Z(R)$. Since the centre of a prime ring is free from zero divisors, we have either $k_1 = 0$ for all $k_1 \in S(R) \cap Z(R)$ or $[h, k] = 0$ for all $h \in (R)$ and $k \in S(R)$. But $S(R) \cap Z(R) \neq (0)$, we conclude that $[h, k] = 0$ for all $h \in H(R)$ and $k \in S(R)$. That is, R is normal. Hence, application of Lemma 2.2.1 yields the required result. On the other hand, if $[x, x^*] = 0$ for all $x \in R$, then R is normal. Hence, R is commutative by Lemma 2.2.1. This proves the lemma. □

Theorem 3.3.1. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If R admits a nonzero left centralizer $T : R \to R$ such that $T([x, x^*]) = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$, then R is commutative.*

Proof. By the given assumption, we have

$$T([x, x^*]) = 0 \text{ for all } x \in R. \tag{3.3.1}$$

Linearizing (3.3.1) and using it, we obtain

$$T([x, y^*] + [y, x^*]) = 0 \text{ for all } x, y \in R. \tag{3.3.2}$$

Replacing y by xx^* in (3.3.2) and using (3.3.1), we get

$$\begin{aligned} 0 &= T([x, xx^*] + [xx^*, x^*]) \\ &= T(x[x, x^*] + [x, x^*]x^*) \\ &= T(x)[x, x^*] + T([x, x^*])x^* \\ &= T(x)[x, x^*] \text{ for all } x \in R. \end{aligned}$$

The last relation forces that

$$T(x)[x, x^*] = 0 \text{ for all } x \in R. \tag{3.3.3}$$

Replacing x by $h + k$, where $h \in H(R)$, $k \in S(R)$, we obtain

$$T(h)[k, h] - T(h)[h, k] + T(k)[k, h] - T(k)[h, k] = 0 \text{ for all } h \in H(R) \text{ and } k \in S(R).$$

This implies that

$$2T(h)[h, k] + 2T(k)[h, k] = 0 \text{ for all } h \in H(R) \text{ and } k \in S(R).$$

Since $char(R) \neq 2$, the above expression forces that

$$T(h)[h, k] + T(k)[h, k] = 0 \text{ for all } h \in H(R) \text{ and } k \in S(R). \tag{3.3.4}$$

Replacing h by $-h$ in (3.3.4), we get

$$T(h)[h, k] - T(k)[h, k] = 0 \text{ for all } h \in H(R) \text{ and } k \in S(R). \tag{3.3.5}$$

Adding (3.3.4) and (3.3.5) and using the fact that $char(R) \neq 2$, we obtain .

$$T(h)[h,k] = 0 \text{ for all } h \in H(R) \text{ and } k \in S(R). \tag{3.3.6}$$

Replacing h by $h+h'$ in (3.3.6), where $h' \in H(R) \cap Z(R)$ we obtain

$$T(h')[h,k] = 0 \text{ for all } h \in H(R),\ h' \in H(R) \cap Z(R) \text{ and } k \in S(R). \tag{3.3.7}$$

Replacing k by h_1k' in (3.3.7), where $h_1 \in H(R)$ and $k' \in S(R) \cap Z(R)$, we get $T(h')[h,h_1]k' = 0$ for all $h, h_1 \in H(R)$, $h' \in H(R) \cap Z(R)$ and $k' \in S(R) \cap Z(R)$. Using the fact that the centre of a prime ring is free from zero divisors and $S(R) \cap Z(R) \neq (0)$, we obtain

$$T(h')[h,h_1] = 0 \text{ for all } h, h_1 \in H(R) \text{ and } h' \in H(R) \cap Z(R). \tag{3.3.8}$$

Now since every $x \in R$ can be represented as $2x = h+k$, where $h \in H(R)$, $k \in S(R)$, in view of equations (3.3.7) and (3.3.8), we obtain $0 = T(h')[h,2x] = 2T(h')[h,x]$. Since $charR \neq 2$, we arrive at $T(h')[h,x] = 0$ for all $x \in R$, $h \in H(R)$ and $h' \in H(R) \cap Z(R)$. Replacing x by yx in the above equation and using it, we get $T(h')y[h,x] = 0$. Using the primeness of R, we have either $T(h') = 0$ for all $h' \in H(R) \cap Z(R)$ or $[h,x] = 0$ for all $x \in R$ and $h \in H(R)$. Suppose $T(h') = 0$ for all $h' \in H(R) \cap Z(R)$. Replacing h' by $(k')^2$, where $k' \in S(R) \cap Z(R)$, we get $T(k')k' = 0$ for all $k' \in S(R) \cap Z(R)$. Since R is prime, the last expression yields that either $T(k') = 0$ for all $k' \in S(R) \cap Z(R)$ or $k' = 0$ for all $k' \in S(R) \cap Z(R)$. In other words $S(R) \cap Z(R)$ is the union of its additive subgroups $A = \{k' \in S(R) \cap Z(R) \mid T(k') = 0\}$ and $B = \{k' \in S(R) \cap Z(R) \mid k' = 0\}$. But a group cannot be the union of two of its proper subgroups hence, either $A = S(R) \cap Z(R)$ or $B = S(R) \cap Z(R)$. Since $S(R) \cap Z(R) \neq (0)$, we are left with $A = S(R) \cap Z(R)$ i.e., $T(k') = 0$ for all $k' \in S(R) \cap Z(R)$. Thus, we have

$$T(h') = 0 \text{ for all } h' \in H(R) \cap Z(R). \tag{3.3.9}$$

$$T(k') = 0 \text{ for all } k' \in S(R) \cap Z(R). \tag{3.3.10}$$

Let $x_1 \in Z(R)$. Since $char(R) \neq 2$, every $x_1 \in Z(R)$ can be represented as $2x_1 =$

$h_1 + k_1$ where $h_1 \in H(R) \cap Z(R)$ and $k_1 \in S(R) \cap Z(R)$. This implies that $T(2x_1) = T(h_1 + k_1) = T(h_1) + T(k_1) = 0$. This implies that $T(x_1) = 0$ for all $x_1 \in Z(R)$. Now $x_1 \in Z(R)$ implies $x_1 y = y x_1$ for all $y \in R$. This yields $T(x_1)y = T(y)x_1$ for all $y \in R$. This give $T(y)x_1 = 0$ for all $x_1 \in Z(R)$ and $y \in R$. Thus the primeness of R yields that either $x_1 = 0$ for all $x_1 \in Z(R)$ or $T(y) = 0$ for all $y \in R$. Which gives a contradiction since $Z(R) \neq 0$ and $T \neq 0$. Thus the only possibility is $[h, x] = 0$ for all $x \in R$ and $h \in H(R)$. That is, R is normal. Hence R is commutative by Lemma 2.2.1. This completes the proof of the theorem. □

Corollary 3.3.1. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If R admits a nonzero left centralizer $T : R \to R$ such that $T([x^*, x]) = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$, then R is commutative.*

If we replace commutator by anti-commutator in Theorem 3.3.1, the corresponding result also holds.

Theorem 3.3.2. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If R admits a nonzero left centralizer $T : R \to R$ such that $T(xox^*) = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$, then R is commutative.*

Proof. We are given that

$$T(xox^*) = 0 \text{ for all } x \in R. \tag{3.3.11}$$

Replacing x by $x + y$ in (3.3.11) and using it, we obtain

$$T(xoy^* + yox^*) = 0 \text{ for all } x, y \in R. \tag{3.3.12}$$

Substituting x^2 for y in (3.3.12) and using (3.3.11), we get

$$\begin{aligned} 0 &= T(xo(x^*)^2 + x^2ox^*) \\ &= T((xox^*)x^* - x^*[x, x^*] + (x^*ox)x - x[x^*, x]) \\ &= -T(x^*)[x, x^*] - T(x)[x^*, x] \\ &= T(x)[x, x^*] - T(x^*)[x, x^*] \text{ for all } x \in R. \end{aligned}$$

The last relation forces that

$$T(x - x^*)[x, x^*] = 0 \text{ for all } x \in R.$$

Replacing x by $h+k$, where $h \in H(R)$ and $k \in S(R)$ and using the fact that $char(R) \neq 2$, we get

$$T(k)[h,k] = 0 \text{ for all } h \in H(R) \text{ and } k \in S(R). \qquad (3.3.13)$$

Substituting $k + k'$ for k, where $k' \in S(R) \cap Z(R)$ in (3.3.13), we find that

$$T(k')[h,k] = 0 \text{ for all } h \in H(R),\ k \in S(R) \text{ and } k' \in S(R) \cap Z(R). \qquad (3.3.14)$$

Replacing h by k_1k_2' in (3.3.14), where $k_1 \in S(R)$ and $k_2' \in S(R) \cap Z(R)$, we get $T(k')[k_1,k]k_2' = 0$ for all $k, k_1 \in S(R)$, $k', k_2' \in S(R) \cap Z(R)$. Using the fact that the centre of a prime ring is free from zero divisors and $S(R) \cap Z(R) \neq (0)$, we obtain

$$T(k')[k_1,k] = 0 \text{ for all } k, k_1 \in S(R) \text{ and } k' \in S(R) \cap Z(R). \qquad (3.3.15)$$

Now since every $x \in R$ can be represented as $2x = h+k$, where $h \in H(R)$, $k \in S(R)$, in view of equations (3.3.14) and (3.3.15), we obtain $0 = T(k')[2x,k] = 2T(k')[x,k]$. Since $charR \neq 2$, we arrive at $T(k')[x,k] = 0$ for all $x \in R$, $k \in S(R)$ and $k' \in S(R) \cap Z(R)$. Replacing x by yx in the above equation and using it, we get $T(k')y[x,k] = 0$. Using the primeness of R, we have either $T(k') = 0$ for all $k' \in S(R) \cap Z(R)$ or $[x,k] = 0$ for all $x \in R$ and $k \in S(R)$. Suppose $T(k') = 0$ for all $k' \in S(R) \cap Z(R)$. Substituting $k'h'$ for k', where $h' \in H(R) \cap Z(R)$ we get $T(k'h') = 0$ that is, $k'T(h') = 0$ for all $h' \in H(R) \cap Z(R)$ and $k' \in S(R) \cap Z(R)$. Again using the fact that the centre of a prime ring is free from zero divisors we have either $k' = 0$ for all $k' \in S(R) \cap Z(R)$ or $T(h') = 0$ for all $h' \in H(R) \cap Z(R)$. But $S(R) \cap Z(R) \neq (0)$. So we have the only possibility that $T(h') = 0$ for all $h' \in H(R) \cap Z(R)$. Therefore, we have

$$T(h') = 0 \text{ for all } h' \in H(R) \cap Z(R). \qquad (3.3.16)$$

$$T(k') = 0 \text{ for all } k' \in S(R) \cap Z(R). \qquad (3.3.17)$$

Henceforth using similar approach as we have used after equations (3.3.9) and (3.3.10) in the proof of Theorem 3.3.1, we get the required result. This finishes the proof of the theorem. □

3.4 The condition $T([x,x^*]) - [x,x^*] = 0$

In the year 1992, Daif and Bell [55] established that a semiprime ring admitting a derivation d such that $d([x,y]) \pm [x,y] = 0$ for all $x, y \in R$ must be commutative. In [14], Argaç generalized this result for a nonzero ideal of a semiprime ring. Recently, Ashraf and Ali [15] studied same result in the setting of prime ring involving left centralizer. Precisely, they proved the following result:

Theorem 3.4.1. *Let R be a prime ring and I be a nonzero ideal of R. Suppose that R admits a nonzero left centralizer T such that $T(x) \neq \pm\, x$ for all $x \in I$. Further, if $T([x,y]) - [x,y] = 0$ for all $x, y \in I$ or $T([x,y]) + [x,y] = 0$ for all $x, y \in I$, then R is commutative.*

The objective of this section is to study the similar problems in the setting of rings with involution involving left centralizers. In fact, we prove the following:

Theorem 3.4.2. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If R admits a nontrivial left centralizer $T : R \to R$ such that $T([x,x^*]) \pm [x,x^*] = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$, then R is commutative.*

Proof. First we consider the case

$$T([x,x^*]) - [x,x^*] = 0 \text{ for all } x \in R. \tag{3.4.1}$$

Linearization of (3.4.1), yields that

$$T([x,y^*]) - [x,y^*] + T([y,x^*]) - [y,x^*] = 0 \text{ for all } x, y \in R. \tag{3.4.2}$$

Replacing y by x^2 in (3.4.2) and using (3.4.1), we obtain

$$T(x^*)[x,x^*] + T(x)[x,x^*] - x^*[x,x^*] - x[x,x^*] = 0 \text{ for all } x \in R.$$

The above relation can be further written as

$$(T(x+x^*) - (x+x^*))[x,x^*] = 0 \text{ for all } x \in R. \tag{3.4.3}$$

Taking $x = h + k$ where $h \in H(R)$ and $k \in S(R)$ in (3.4.3) and using the fact that

$char(R) \neq 2$, we get

$$(T(h) - h)[h, k] = 0 \text{ for all } h \in H(R), k \in S(R). \tag{3.4.4}$$

Let $h' \in H(R) \cap Z(R)$. Replacing h by $h + h'$ in (3.4.4), we obtain

$$(T(h') - h')[h, k] = 0 \text{ for all } h' \in H(R) \cap Z(R),\ h \in H(R) \text{ and } k \in S(R). \tag{3.4.5}$$

Replacing k by h_1k' in (3.4.5), where $h_1 \in H(R)$ and $k' \in S(R) \cap Z(R)$, we get $(T(h') - h')[h, h_1]k' = 0$ for all $h, h_1 \in H(R)$, $h' \in H(R) \cap Z(R)$ and $k' \in S(R) \cap Z(R)$. Using the fact that the centre of a prime ring is free from zero divisors and $S(R) \cap Z(R) \neq (0)$, we obtain

$$(T(h') - h')[h, h_1] = 0 \text{ for all } h, h_1 \in H(R) \text{ and } h' \in H(R) \cap Z(R). \tag{3.4.6}$$

Now since every $x \in R$ can be represented as $2x = h + k$, where $h \in H(R)$, $k \in S(R)$, in view of equations (3.4.5) and (3.4.6), we obtain $0 = (T(h') - h')[h, 2x] = 2(T(h') - h')[h, x]$. Since $charR \neq 2$, we arrive at $(T(h') - h')[h, x] = 0$ for all $x \in R$, $h \in H(R)$ and $h' \in H(R) \cap Z(R)$. Replacing x by yx in the above equation and using it, we get $(T(h') - h')y[h, x] = 0$. Using the primeness of R, we have either $T(h') - h' = 0$ for all $h' \in H(R) \cap Z(R)$ or $[h, x] = 0$ for all $x \in R$ and $h \in H(R)$. Suppose $T(h') = h'$ for all $h' \in H(R) \cap Z(R)$. Replacing h' by $(k')^2$ where $k' \in S(R) \cap Z(R)$, we have $T((k')^2) - (k')^2 = 0$. This implies $(T(k') - k')k' = 0$ for all $k' \in S(R) \cap Z(R)$. Using the primeness of R, we have either $k' = 0$ for all $k' \in S(R) \cap Z(R)$ or $T(k') = k'$ for all $k' \in S(R) \cap Z(R)$. In other words $S(R) \cap Z(R)$ is the union of its additive subgroups $A = \{k' \in S(R) \cap Z(R) \mid T(k') = k'\}$ and $B = \{k' \in S(R) \cap Z(R) \mid k' = 0\}$. But a group cannot be the union of two of its proper subgroups hence, either $A = S(R) \cap Z(R)$ or $B = S(R) \cap Z(R)$. But since $S(R) \cap Z(R) \neq (0)$, we are left with $A = S(R) \cap Z(R)$ that is, $T(k') = k'$ for all $k' \in S(R) \cap Z(R)$. Thus we have

$$T(h') = h' \text{ for all } h' \in H(R) \cap Z(R). \tag{3.4.7}$$

$$T(k') = k' \text{ for all } k' \in S(R) \cap Z(R). \tag{3.4.8}$$

Let $x_1 \in Z(R)$. Since $char(R) \neq 2$, every $x_1 \in Z(R)$ can be represented as $2x_1 = h_1+k_1$ where $h_1 \in H(R) \cap Z(R)$ and $k_1 \in S(R) \cap Z(R)$. This implies that $T(2x_1) = T(h_1 + k_1) = T(h_1) + T(k_1) = h_1 + k_1 = 2x_1$. Thus we obtain

$$T(x_1) = x_1 \text{ for all } x_1 \in Z(R). \tag{3.4.9}$$

But $x_1 \in Z(R)$ implies $x_1y = yx_1$ for all $y \in R$. This yields $T(x_1)y = T(y)x_1$ for all $y \in R$. Using (3.4.9) we obtain $(T(y) - y)x_1 = 0$ for all $x_1 \in Z(R)$ and $y \in R$. Using the primeness of R we have $x_1 = 0$ for all $x_1 \in Z(R)$ or $T(y) = y$ for all $y \in R$. Which gives a contradiction, since $Z(R) \neq (0)$ and T is nontrivial. Thus the only possibility is that, $[h, k] = 0$ for all $h \in H(R)$ and $k \in S(R)$ and hence R is normal. In view of Lemma 2.2.1, we conclude that R is commutative.

By the same argument, we obtain the similar conclusion in the case $T([x, x^*]) + [x, x^*] = 0$ for all $x \in R$. This proves the theorem completely. □

Corollary 3.4.1. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If R admits a nontrivial left centralizer $T : R \to R$ such that $T([x^*, x]) \pm [x^*, x] = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$, then R is commutative.*

Theorem 3.4.3. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If R admits a nontrivial left centralizer $T : R \to R$ such that $T(xox^*) \pm (xox^*) = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$, then R is commutative.*

Proof. First we consider the case

$$T(xox^*) - (xox^*) = 0 \text{ for all } x \in R. \tag{3.4.10}$$

Linearizing the above relation, we get

$$T(xoy^*) - xoy^* + T(yox^*) - yox^* = 0 \text{ for all } x, y \in R. \tag{3.4.11}$$

Replacing y by x^2 in (3.4.11) and using (3.4.10), we obtain

$$-T(x^*)[x, x^*] + T(x)[x, x^*] + x^*[x, x^*] - x[x, x^*] = 0 \text{ for all } x \in R.$$

This can be further written as

$$(T(x-x^*)-(x-x^*))[x,x^*]=0 \text{ for all } x \in R. \tag{3.4.12}$$

Replacing x by $h+k$, where $h \in H(R)$ and $k \in S(R)$ and using the fact that $char(R) \neq 2$, we get

$$(T(k)-k)[h,k]=0 \text{ for all } h \in H(R),\ k \in S(R). \tag{3.4.13}$$

Let $k^{'} \in S(R) \cap Z(R)$. Replacing k by $k+k^{'}$ in (3.4.13), we obtain

$$(T(k^{'})-k^{'})[h,k]=0 \text{ for all } k^{'} \in S(R) \cap Z(R),\ h \in H(R) \textit{ and } k \in S(R). \tag{3.4.14}$$

Replacing h by $k_1k_2^{'}$ in (3.4.14), where $k_1 \in S(R)$ and $k_2^{'} \in S(R) \cap Z(R)$, we get $(T(k^{'})-k^{'})[k_1,k]k_2^{'}=0$ for all $k, k_1 \in S(R)$, $k^{'}, k_2^{'} \in S(R) \cap Z(R)$. Using the fact that the centre of a prime ring is free from zero divisors and $S(R) \cap Z(R) \neq (0)$, we obtain

$$(T(k^{'})-k^{'})[k_1,k]=0 \text{ for all } k, k_1 \in S(R) \text{ and } k^{'} \in S(R) \cap Z(R). \tag{3.4.15}$$

Now since every $x \in R$ can be represented as $2x=h+k$, where $h \in H(R)$, $k \in S(R)$, in view of equations (3.4.14) and (3.4.15), we obtain $0=(T(k^{'})-k^{'})[2x,k]=2(T(k^{'})-k^{'})[x,k]$. Since $charR \neq 2$, we arrive at $(T(k^{'})-k^{'})[x,k]=0$ for all $x \in R$, $k \in S(R)$ and $k^{'} \in S(R) \cap Z(R)$. Replacing x by yx in the above equation and using it, we get $(T(k^{'})-k^{'})y[x,k]=0$. Using the primeness of R, we have either $T(k^{'})=k^{'}$ for all $k^{'} \in S(R) \cap Z(R)$ or $[x,k]=0$ for all $x \in R$ and $k \in S(R)$. Suppose $T(k^{'})=k^{'}$ for all $k^{'} \in S(R) \cap Z(R)$. Replacing $k^{'}$ by $h^{'}k^{'}$, where $h^{'} \in H(R) \cap Z(R)$, we get $T(h^{'})k^{'}=h^{'}k^{'}$. This implies $(T(h^{'})-h^{'})k^{'}=0$ for all $h^{'} \in H(R) \cap Z(R)$ and $k^{'} \in S(R) \cap Z(R)$. Again using the fact that the centre of a prime ring is free from zero divisors we have $T(h^{'})=h^{'}$ for $h^{'} \in H(R) \cap Z(R)$ or $k^{'}=0$ for all $k^{'} \in S(R) \cap Z(R)$. But since $S(R) \cap Z(R) \neq (0)$ we have $T(h^{'})=h^{'}$ for all $h^{'} \in H(R) \cap Z(R)$. Therefore we find that

$$T(h^{'})=h^{'} \text{ for all } h^{'} \in H(R) \cap Z(R). \tag{3.4.16}$$

$$T(k^{'})=k^{'} \text{ for all } k^{'} \in S(R) \cap Z(R). \tag{3.4.17}$$

Hence using the same approach as we have used after equations (3.4.7) and (3.4.8) in the proof of the Theorem 3.4.2, we get the required result.

By the same argument, we obtain the similar conclusion in the case $T(xox^*) + (xox^*) = 0$ for all $x \in R$. This proves the theorem. $\square$

Theorem 3.4.4. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If R admits a nontrivial left centralizer $T : R \to R$ such that $T([x, x^*]) \pm (xox^*) = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$, then R is commutative.*

Proof. First we consider the case

$$T([x, x^*]) - (xox^*) = 0 \text{ for all } x \in R. \tag{3.4.18}$$

Linearizing the above relation, we get

$$T([x, y^*]) - xoy^* + T([y, x^*]) - yox^* = 0 \text{ for all } x, y \in R. \tag{3.4.19}$$

Replacing y by x^2 in (3.4.19) and using (3.4.18), we obtain

$$T(x^*)[x, x^*] + T(x)[x, x^*] + x^*[x, x^*] + x[x, x^*] = 0 \text{ for all } x \in R.$$

This can be further written as

$$(T(x + x^*) + (x + x^*))[x, x^*] = 0 \text{ for all } x \in R. \tag{3.4.20}$$

Replacing x by $h+k$, where $h \in H(R)$ and $k \in S(R)$ and using the fact that $char(R) \neq 2$, we get

$$(T(h) + h)[h, k] = 0 \text{ for all } h \in H(R),\ k \in S(R). \tag{3.4.21}$$

Henceforth, using the similar approach with necessary variations as we have used after equation (3.4.4) in the proof of the Theorem 3.4.2, we get R is normal. Further in view of Lemma 2.2.1, we get the commutativity of R.

By the same argument, we obtain the similar conclusion in case $T([x, x^*]) + (xox^*) = 0$ for all $x \in R$. This completes the proof of theorem. $\square$

Corollary 3.4.2. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If*

R admits a nontrivial left centralizer $T : R \to R$ such that $T([x^*, x]) \pm (xox^*) = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$, then R is commutative.

We end this chapter with the following example which shows that the condition $S(R) \cap Z(R) \neq (0)$ in the hypothesis of Theorem 3.3.1 and Theorem 3.4.2 is not superfluous.

Example 3.4.1. Let

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c, d \in Z \right\}.$$

Of course R with matrix addition and matrix multiplication is a prime ring. Define mappings $T : R \longrightarrow R$, and $* : R \longrightarrow R$ such that

$$T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \qquad a, b, c, d \in Z.$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad a, b, c, d \in Z.$$

Then $x^* = x$ for all $x \in Z(R)$, hence $Z(R) \subseteq H(R)$. This implies that $S(R) \cap Z(R) = (0)$. Moreover, T is a nonzero left centralizer and the following conditions: (i) $T([x, x^*]) = 0$, (ii) $T([x, x^*]) \pm [x, x^*] = 0$ for all $x \in R$, are satisfied. However, R is not commutative. Hence, the condition $S(R) \cap Z(R) \neq (0)$ is crucial in Theorem 3.3.1 and Theorem 3.4.2.

CHAPTER-4

On Jordan Left *-Centralizers in Rings with Involution and their Applications

Chapter 4

On Jordan Left $*$-Centralizers in Rings with Involution and their Applications

4.1 Introduction

Following [146], an additive mapping $T : R \to R$ is called a left centralizer in case $T(xy) = T(x)y$ holds for all $x, y \in R$. An additive mapping $T : R \to R$ is said to be a reverse left centralizer if $T(xy) = T(y)x$ holds for all $x, y \in R$. The definition of a reverse right centralizer should be self-explanatory. For a semiprime ring R, all left centralizers are of the form $T(x) = qx$ for all $x \in R$, where q is an element of Martindale right ring of quotients $Q_r(R)$ of R (see [37, Chapter 2] for details). In case if R has an identity element, then $T : R \to R$ is a left centralizer if and only if T is of the form $T(x) = ax$ for all $x \in R$ and some fixed element $a \in R$. In case $T : R \to R$ is a two-sided centralizer, where R is a semiprime ring with extended centoid C, then there exists an element $\lambda \in C$ such that $T(x) = \lambda x$ for all $x \in R$ (viz.; [37, Theorem 2.3.2]). An additive mapping $T : R \to R$ is called a Jordan left centralizer (resp. Jordan right centralizer) if $T(x^2) = T(x)x$ (resp. $T(x^2) = xT(x)$) holds for all $x \in R$. Clearly, every left centralizer (resp. right centralizer) on a ring R is a Jordan left centralizer on R. But the converse of this statement need not be true in general. However, in case of prime ring of characteristic different from two both concepts coincide (see [49, Proposition 2.5]). Further, Zalar [146] proved this result in the setting of semiprime ring of characteristic different from two. More related results on centralizers in rings and algebras can be looked in [5, 62, 63, 106, 138, 139, 140, 141, 142, 143], where

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further references can be found.

Let R be a ring with involution $*$. Motivated by the definitions of left (resp. right) centralizer and Jordan left (resp. right) centralizer in rings, Ali and Fošner in [4] introduced the notion of left (resp. right) $*$-centralizer and Jordan left (resp. right) $*$-centralizer as follows: an additive mapping $T : R \longrightarrow R$ is said to be a left $*$-centralizer (resp. Jordan left $*$-centralizer) if $T(xy) = T(x)y^*$(resp. $T(x^2) = T(x)x^*$) holds for all $x, y \in R$. The definition of right $*$-centralizer (resp. Jordan right $*$-centralizer) should be self explanatory. An additive mapping $T : R \longrightarrow R$ is said to be a reverse left $*$-centralizer if $T(xy) = T(y)x^*$ is fulfilled for all $x, y \in R$. Reverse right $*$-centralizer is defined in a similar way. An additive mapping $T : R \longrightarrow R$ is called a $*$-centralizer (resp. reverse $*$-centralizer) if T is both a left and a right $*$-centralizer (resp. reverse left and right $*$-centralizer). Note that for some fixed element $a \in R$, the mapping $x \mapsto ax^*$ is a Jordan left $*$-centralizer and $x \mapsto x^*a$ is a Jordan right $*$-centralizer on R. Clearly, every reverse left $*$-centralizer on a ring R is a Jordan left $*$-centralizer. Thus, it is natural to question that whether the converse of above statement is true. In Section 4.2, it is shown that the answer to this question is affirmative if the underlying $*$-ring R is 2-torsion free semiprime ring. Further, we establish a result concerning additive mapping $T : R \rightarrow R$ satisfying the relation $T(x^{m+n+1}) = (x^*)^n T(x)(x^*)^m$ for all $x \in R$, where m and n are positive integers. Moreover, some nice characterization of $*$-centralizers in prime and semiprime rings are also given.

According to [45], an additive mapping $d : R \rightarrow R$ is said to be a Jordan $*$-derivation of R if $d(x^2) = d(x)x^* + xd(x)$ for all $x \in R$. The notion of Jordan $*$-derivations appears first time in [45], where some algebraic properties are studied. In [49], Brešar and Zalar obtained a representation of Jordan $*$-derivations in terms of left and right centralizers on the algebra of compact operators on Hilbert spaces. Further, Brešar and Vukman [45] characterized normal rings by using the theory of Jordan $*$-derivations in prime rings with involution. In Section 4.2, we generalize the above mentioned result for Jordan left $*$-centralizers.

In Section 4.3, we study the result obtained by Vukman [131, Theorem 4] in the setting of rings with involution by replacing left centralizer with Jordan left $*$-centralizer. In fact, we prove the following result: Let R be a noncommutative 2-torsion free semiprime ring with involution $*$ and $S, T : R \longrightarrow R$ be Jordan left $*$-centralizers.

Suppose that

$$[S(x), T(x)]S(x) - S(x)[S(x), T(x)] = 0 \text{ holds for all } x \in R.$$

Then $[S(x), T(x)] = 0$ for all $x \in R$. Moreover, if R is a prime ring and $S \neq 0$ $(T \neq 0)$, then there exists $\lambda \in C$ such that $T = \lambda S$ $(S = \lambda T)$. Moreover, we also prove the above result by replacing Lie product with the Jordan product and obtain the similar conclusion.

In Section 4.4, we present some applications of the theory of $*$-centralizers in rings with involution.

We shall restrict our attention on left $*$-centralizers (resp. Jordan left $*$-centralizers) since all results presented in this chapter are also true for right $*$-centralizers (resp. Jordan right $*$-centralizers) because of left and right symmetry.

4.2 On $*$-centralizers in rings

Perhaps, it was Brešar and Zalar [49] who first introduced the concept of Jordan centralizers of a ring R, and established that on a prime ring of characteristic different from two every Jordan left centralizer (resp. Jordan right centralizer) is a left centralizer (resp. right centralizer) on R [49, Proposition 2.5]. Further, in [146] Zalar generalized the above mentioned result for semprime ring (without involution). In view of this result, it is natural to question that whether the above result is true in case of ring with involution. In the present section, it is shown that the answer to this question is affirmative if the underlying $*$-ring R is a 2-torsion free semiprime. In fact, we prove the following result:

Proposition 4.2.1. *Let R be a 2-torsion free semiprime ring with involution $*$ and $T : R \to R$ be an additive map which satisfies $T(x^2) = T(x)x^*$ for all $x \in R$. Then, T is a reverse left $*$-centralizer that is, $T(xy) = T(y)x^*$ for all $x,\ y \in R$.*

Proof. By the assumption, we have $T(x^2) = T(x)x^*$ for all $x \in R$. Applying involution $*$ both sides to the above expression, we obtain

$$(T(x^2))^* = x(T(x))^* \text{ for all } x \in R.$$

Define a new map $S : R \to R$ such that $S(x) = (T(x))^*$ for all $x \in R$. Then we see that

$$\begin{aligned} S(x^2) &= (T(x^2))^* \\ &= (T(x)x^*)^* \\ &= x(T(x))^* \\ &= xS(x) \text{ for all } x \in R. \end{aligned}$$

Hence, we obtain $S(x^2) = xS(x)$ for all $x \in R$. Thus, S is a Jordan right centralizer on R. In view of [146, Proposition 1.4], S is a right centralizer that is, $S(xy) = xS(y)$ for all $x, y \in R$. This implies that $(T(xy))^* = x(T(y))^*$ for all $x, y \in R$. By applying involution to the both sides of the last relation, we find that $T(xy) = T(y)x^*$ for all $x, y \in R$. This completes the proof of the proposition. $\square$

Our next result is motivated by the work of Hvala [79, Lemma 2].

Proposition 4.2.2. *Let R be a prime ring with involution $*$ and let $T : R \longrightarrow R$ be a nonzero additive map satisfying $T(xy) = T(x)y^*$ for all $x, y \in R$. Then R is commutative and there exists $\lambda \in C$ such that $T(x) = \lambda x^*$ for all $x \in R$.*

Proof. In view of our hypothesis, we have $T(xy) = T(x)y^*$ for all $x, y \in R$. This gives $T(xyz) = T(x)(yz)^* = T(x)z^*y^*$. On the other hand $T(xyz) = T(xy)z^* = T(x)y^*z^*$. Combining the above two expressions, we obtain $T(x)[y^*, z^*] = 0$ for all $x, y, z \in R$. This implies $T(R)R[R, R] = (0)$. Since T is nonzero and R is prime, we see that R is commutative. Consequently, a map $g : R \to R$ defined by $g(x) = T(x)^*$ is an R-module homomorphism (i.e., a centralizer). Hence, there exists $\mu \in C$ such that $g(x) = \mu x$ and so $T(x) = \lambda x^*$ for all $x \in R$, where $\lambda := \mu^*$. This proves the proposition completely. $\square$

Proposition 4.2.3. *Let R be a semiprime ring with involution $*$ and let $T : R \longrightarrow R$ be an additive map satisfying $T(xy) = T(y)x^*$ for all $x, y \in R$. Then there exists $q \in Q_r(R)$ such that $T(x) = qx^*$ for all $x \in R$.*

Proof. Note that a map $g : R \to R$ defined by $g(x) = T(x^*)$ is a right R-module homomorphism (i.e., a left centralizer). Hence, there exists $q \in Q_r(R)$ such that $g(x) = qx$ for all $x \in R$. Thus, $T(x) = qx^*$ for all $x \in R$. $\square$

Proposition 4.2.4. *Let R be a 2-torsion free semiprime ring with involution $*$. If $T : R \to R$ is an additive map such that $T(x^2) = T(x)x^*$ for all $x \in R$, then there exists $q \in Q_r(R)$ such that $T(x) = qx^*$ for all $x \in R$.*

Proof. Since $T(x^2) = T(x)x^*$ for all $x \in R$. Therefore, in view of Proposition 4.2.1 $T(xy) = T(y)x^*$ for all $x, y \in R$. Now, Proposition 4.2.3 yields that $T(x) = qx^*$ for all $x \in R$, where $q \in Q_r(R)$. □

Theorem 4.2.1. *Let m and n be positive integers, and let R be a prime ring with involution $*$ such that $char(R) = 0$ or $m + n + 1 \leq char(R)$. If $T : R \to R$ is an additive map satisfying the relation $T(x^{m+n+1}) = (x^*)^n T(x)(x^*)^m$ for all $x \in R$, then $T(xy) = T(y)x^* = y^*T(x)$ for all $x,\ y \in R$ that is, T is a reverse $*$-centralizer on R.*

Proof. By the given hypothesis, we have $T(x^{m+n+1}) = (x^*)^n T(x)(x^*)^m$ for all $x \in R$. Invoking involution both sides to the last expression, we obtain $T(x^{m+n+1})^* = x^m T(x)^* x^n$ for all $x \in R$. Define a new map $S : R \to R$ such that $S(x) = T(x)^*$ for all $x \in R$. Then S is additive, since T is additive. Therefore, we have $S(x^{m+n+1}) = x^m S(x) x^n$ for all $x \in R$. Hence in view of Lemma 1.3.8, we are forced to conclude that S is a two sided-centralizer that is, $S(xy) = xS(y) = S(x)y$ for all $x, y \in R$. This implies that $(T(xy))^* = x(T(y))^* = (T(x))^* y$ for all $x, y \in R$. Again applying involution both sides to the last expression, we find that $T(xy) = T(y)x^* = y^*T(x)$ for all $x, y \in R$. The theorem is now completely proved. □

Following are the immediate consequences of Theorem 4.2.1:

Corollary 4.2.1. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$ and $T : R \to R$ be an additive map which satisfies $T(x^3) = x^*T(x)x^*$ for all $x \in R$. Then $T(xy) = T(y)x^* = y^*T(x)$ for all $x,\ y \in R$ that is, T is a reverse $*$-centralizer on R.*

Corollary 4.2.2. *Let m and n be positive integers, and let R be a prime ring with involution $*$ such that $char(R) = 0$ or $m + n + 1 \leq char(R)$. If $T : R \to R$ is an additive map which satisfies the relation $T(x^{m+n+1}) = (x^*)^n T(x)(x^*)^m$ for all $x \in R$, then there exists $q \in Q_r(R)$ such that $T(x) = qx^*$ for all $x \in R$.*

Proof. In view of Theorem 4.2.1, we have $T(xy) = T(y)x^*$ for all $x, y \in R$. Now, Proposition 4.2.3 yields that $f(x) = qx^*$ for all $x \in R$, where $q \in Q_r(R)$. □

The next theorem characterizes normal rings among all noncommutative prime $*$-rings of characteristic different from two.

Theorem 4.2.2. *Let R be a noncommutative prime ring with involution $*$ such that $char(R) \neq 2$. Then the following conditions are mutually equivalent:*

(i) R is normal

(ii) there exists a nonzero commuting Jordan left $*$-centralizer T on R.

Proof. Suppose R is a normal ring. Then the mapping $x \mapsto x^*$ is a commuting nonzero Jordan left $*$-centralizer on R. Now suppose (ii) holds, we have to prove R is normal. By Lemma 1.3.5, there exists $\mu \in C$ and a map $\nu : R \to C$ such that

$$T(x) = \mu x + \nu(x) \text{ for all } x \in R.$$

On the other hand, it follows from the Proposition 4.2.4, $T(x) = qx^*$ for all $x \in R$, where $q \in Q_r(R)$. Thus, we have

$$qx^* - \mu x \in C \text{ for all } x \in R.$$

Since the identity involves involution, so it is a functional identity or the so-called g-identity (see [37, Chapter 6]). In view of Lemma 1.3.2, we conclude that $qx^* - \mu x \in C$ for all $x \in Q_s(R)$, the symmetric ring of quotients. Note that $Q_s(R)$ has the identity element 1. Replacing x by 1 in the above expression, we see that $q - \mu \in C$. This implies that $[q, y] = 0$ for all $y \in Q_s(R)$. Thus,

$$T(x) = \lambda x^* \text{ for all } x \in R,$$

where $\lambda = q \in C$. Since $T \neq 0$, it follows that $\lambda \neq 0$. Thus we conclude that $0 = [T(x), x] = [\lambda x^*, x] = \lambda[x^*, x]$ for all $x \in R$. The primeness of R yields that R is normal. This proves the theorem completely. □

The above theorem has the following interesting consequence:

Corollary 4.2.3. *Let R be a non-commutative prime ring with involution $*$ such that $char(R) \neq 2$. If T is a nonzero Jordan left $*$-centralizer on R such that $[T(x), x] = 0$*

for all $x \in R$, then there exists $\lambda \in C$, the extended centroid of R such that $T(x) = \lambda x^*$ for all $x \in R$.

Theorem 4.2.3. *Let R be a noncommutative prime ring with involution $*$ such that $char(R) \neq 2$. If T_1 and T_2 are two nonzero Jordan left $*$-centralizers on R such that $T_1(x)x - xT_2(x) = 0$ for all $x \in R$, then R is normal.*

Proof. By the given hypothesis, we have

$$T_1(x)x - xT_2(x) = 0 \text{ for all } x \in R. \tag{4.2.1}$$

On linearizing (4.2.1), we get

$$T_1(x)y + T_1(y)x - xT_2(y) - yT_2(x) = 0 \text{ for all } x, y \in R. \tag{4.2.2}$$

Replacing y by yx in (4.2.2) and using Proposition 4.2.1, we arrive at

$$T_1(x)yx + T_1(x)y^*x - xT_2(x)y^* - yxT_2(x) = 0 \text{ for all } x, y \in R. \tag{4.2.3}$$

Using (4.2.1) in (4.2.3), we obtain $T_1(x)yx + T_1(x)y^*x - T_1(x)xy^* - yT_1(x)x = 0$ for all $x, y \in R$. This can be further written as $[T_1(x), y]x + T_1(x)[y^*, x] = 0$ for all $x, y \in R$. In view of Proposition 4.2.4, we conclude that $T_1(x) = q_1x^*$ for all $x \in R$, where $q_1 \in Q_r(R)$. Thus $[q_1x^*, y]x + q_1x^*[y^*, x] = 0$ for all $x \in R$. Since the above identity is a g-identity (see [37, Chapter 6]). In view of Lemma 1.3.2, we are force to conclude that $[q_1x^*, y]x + q_1x^*[y^*, x] = 0$ for all $x \in Q_s(R)$, the symmetric ring of quotients. Note that $Q_s(R)$ has the identity element 1. Replacing x by 1 in the above expression, we see that $[q_1, y] = 0$ for all $y \in Q_s(R)$. Thus,

$$T_1(x) = \lambda x^* \text{ for all } x \in R,$$

where $\lambda = q_1 \in C$. Since $T_1 \neq 0$, it follows that $\lambda_1 \neq 0$. Also $T_2(x) = q_2x^*$, by Proposition 4.2.4. Hence from (4.2.1), $\lambda x^*x - xq_2x^* = 0$ for all $x \in R$. Since the above identity is a g-identity. Thus by Lemma 1.3.2, we obtain $\lambda x^*x - xq_2x^* = 0$ for all $x \in Q_s(R)$, the symmetric ring of quotients. Replacing x by 1 in the above expression, we see that $\lambda = q_2$. Thus $T_2(x) = T_1(x) = \lambda x^*$ for all $x \in R$, where $0 \neq \lambda \in C$. Hence, we conclude that $0 = T_1(x)x - xT_2(x) = \lambda x^*x - x\lambda x^* = \lambda(x^*x - xx^*)$ for all $x \in R$.

Since R is prime, the last expression yields that R is normal. This proves the theorem completely. $\square$

As an immediate consequence of Theorem 4.2.3, we have the following corollary:

Corollary 4.2.4. *Let R be a noncommutative prime ring with involution $*$ such that $char(R) \neq 2$. If T_1, T_2 are two nonzero Jordan left $*$-centralizers on R such that $T_1(x)x - xT_2(x) = 0$ for all $x \in R$, then there exists $\lambda \in C$, the extended centroid of R such that $T_1(x) = T_2(x) = \lambda x^*$ for all $x \in R$.*

It would be interesting to know whether Theorem 4.2.2 and Theorem 4.2.3 hold in the case of arbitrary rings. Following examples justify this fact:

Example 4.2.1. *Let F be a field and $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \Big| \, a, b, c \in F \right\}$. Define mappings $T : R \longrightarrow R$, and $* : R \longrightarrow R$ such that $T \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & c & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}$. It is easy to verify that T satisfies all the requirements of Theorem 4.2.2. However, R is not normal.*

Example 4.2.2. *In the above Example 4.2.1, Define mappings $T_1, T_2 : R \longrightarrow R$, such that $T_1 \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $T_2 \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. It is easy to verify that T_1 and T_2 satisfy all the requirements of Theorem 4.2.3. However, R is not normal.*

4.3 On Jordan left $*$-centralizers in rings

During the last few decades, there has been ongoing interest concerning the left centralizers (resp. Jordan left centralizers) on prime and semiprime rings. Recently, many authors viz; [5, 62, 63, 121, 133, 138, 139, 140, 141, 142, 143] have obtained some interesting results in rings and algebras. In [131, Theorem 4], Vukman proved that if R is a noncommutative 2-torsion free semiprime ring and $S, T : R \to R$ are left centralizers such that $[S(x), T(x)]S(x) + S(x)[S(x), T(x)] = 0$ for all $x \in R$, then $[S(x), T(x)] = 0$ for all $x \in R$. In case R is a prime ring and $S \neq 0$ ($T \neq 0$), then there exists $\lambda \in C$ such that $T = \lambda S$ ($S = \lambda T$). The intent of this section is to study similar types of problems in the setting of rings with involution by replacing left centralizer with Jordan left $*$-centralizer.

We begin with the following:

Lemma 4.3.1. *Let R be a noncommutative prime ring with involution $*$ and let $T : R \longrightarrow R$ be a Jordan left $*$-centralizer on R. If $T(x) \in Z(R)$ for all $x \in R$, then $T = 0$.*

Proof. By the hypothesis, we have $[T(x), y] = 0$ for all $x, y \in R$. Substituting x^2 for x in the above relation, we obtain

$$\begin{aligned} 0 &= [T(x^2), y] \\ &= [T(x)x^*, y] \\ &= [T(x), y]x^* + T(x)[x^*, y] \text{ for all } x, y \in R. \end{aligned}$$

In view of our hypothesis, the last expression yields that $T(x)[x^*, y] = 0$ for all x, $y \in R$. Since the centre of a prime ring is free from zero divisors, either $T(x) = 0$ or $[x^*, y] = 0$. Let $A = \{x \in R \mid T(x) = 0\}$ and $B = \{x \in R \mid [x^*, y] = 0 \text{ for all } y \in R\}$. It can be easily seen that A and B are two additive subgroups of R whose union is R and hence by Brauer's trick, we get $A = R$ or $B = R$. If $B = R$, then R is commutative, which gives a contradiction. Thus the only possibility remains that $A = R$. That is, $T(x) = 0$ for all $x \in R$. This finishes the proof. □

Proposition 4.3.1. *Let R be a noncommutative prime ring with involution $*$ and let $S, T : R \longrightarrow R$ be Jordan left $*$-centralizers. Suppose that $[S(x), T(x)] = 0$ holds for all $x \in R$. If $T \neq 0$, then there exists $\lambda \in C$ such that $S = \lambda T$.*

Proof. By Proposition 4.2.1, we conclude that S and T are reverse left $*$-centralizers on R. In view of the hypothesis, we have

$$[S(x), T(x)] = 0 \text{ for all } x \in R. \tag{4.3.1}$$

Linearizing (4.3.1), we get

$$[S(x), T(x)] + [S(x), T(y)] + [S(y), T(y)] + [S(y), T(x)] = 0 \tag{4.3.2}$$

for all $x, y \in R$. Using (4.3.1) in (4.3.2), we have

$$[S(x), T(y)] + [S(y), T(x)] = 0 \text{ for all } x,\ y \in R. \tag{4.3.3}$$

Replacing x by zx in (4.3.3), we obtain

$$[S(x), T(y)]z^* + S(x)[z^*, T(y)] + [S(y), T(x)]z^* + T(x)[S(y), z^*] = 0 \tag{4.3.4}$$

for all $x, y, z \in R$. Application of (4.3.3) yields that

$$S(x)[z^*, T(y)] + T(x)[S(y), z^*] = 0 \text{ for all } x, y, z \in R. \tag{4.3.5}$$

Replacing x by wx in (4.3.5), we get

$$S(x)w^*[z^*, T(y)] + T(x)w^*[S(y), z^*] = 0 \text{ for all } x,\ y,\ z,\ w \in R. \tag{4.3.6}$$

Replacing w by w^* and z by z^* in (4.3.6), we obtain

$$S(x)w[z, T(y)] + T(x)w[S(y), z] = 0 \text{ for all } x,\ y,\ z,\ w \in R. \tag{4.3.7}$$

It follows from Lemma 4.3.1 that there exist $y, z \in R$ such that $[T(y), z] \neq 0$, since $T \neq 0$. In view of Lemma 1.3.10 and from relation (4.3.7), we conclude that $S(x) = \lambda(x)T(x)$, where $\lambda(x)$ is from C. Thus, the relation (4.3.7) forces that

$$\begin{aligned} 0 &= \lambda(x)T(x)w[T(y), z] - T(x)w[\lambda(y)T(y), z] \\ &= \lambda(x)T(x)w[T(y), z] - T(x)w\lambda(y)[T(y), z] \\ &= (\lambda(x) - \lambda(y))T(x)w[T(y), z] \text{ for all } x, y, z, w \in R. \end{aligned}$$

Since R is a prime ring, the above expression yields that either $(\lambda(x) - \lambda(y))T(x) = 0$ or $[T(y), z] = 0$. Since $[T(y), z] \neq 0$, we have $(\lambda(x) - \lambda(y))T(x) = 0$ for all $x, y \in R$. This implies that $\lambda(x)T(x) = \lambda(y)T(x)$ for all $x, y \in R$. This gives $S(x) = \lambda(y)T(x)$ for all $x, y \in R$, as desired. $\square$

If we replace the commutator by anti-commutator in Proposition 4.3.1, the corresponding result also holds.

Proposition 4.3.2. *Let R be a noncommutative prime ring with involution $*$ and let $S,T : R \longrightarrow R$ be Jordan left $*$-centralizers. Suppose that $S(x)oT(x) = 0$ holds for all $x \in R$. If $T \neq 0$, then there exists $\lambda \in C$ such that $S = \lambda T$.*

Proof. By the assumption, we have

$$S(x) \circ T(x) = 0 \text{ for all } x \in R. \tag{4.3.8}$$

Replacing x by $x + y$ in (4.3.8), we obtain

$$S(x) \circ T(x) + S(x) \circ T(y) + S(y) \circ T(x) + S(y) \circ T(y) = 0 \tag{4.3.9}$$

for all $x, y \in R$. Using (4.3.8) in (4.3.9), we get

$$S(x) \circ T(y) + S(y) \circ T(x) = 0 \text{ for all } x,\ y \in R. \tag{4.3.10}$$

Substituting zy for y in (4.3.10), we get $S(x)oT(zy)+S(zy)oT(x) = 0$ for all $x, y, z \in R$. Application of Propsition 4.2.1 yields that

$$\begin{aligned} 0 &= S(x) \circ T(zy) + S(zy) \circ T(x) \\ &= S(x) \circ (T(y)z^*) + T(x) \circ (S(y)z^*) \\ &= (S(x) \circ T(y))z^* - T(y)[S(x), z^*] + (T(x) \circ S(y))z^* - S(y)[T(x), z^*]. \end{aligned}$$

In view of (4.3.10), the above expression yields that

$$T(y)[S(x), z^*] + S(y)[T(x), z^*] = 0 \text{ for all } x, y, z \in R. \tag{4.3.11}$$

Replacing y by wy in (4.3.11), we obtain

$$T(y)w^*[S(x), z^*] + S(y)w^*[T(x), z^*] = 0 \text{ for all } x,\ y,\ z,\ w \in R. \tag{4.3.12}$$

Replacing w by w^* and z by z^* in (4.3.13), we get

$$T(y)w[S(x), z] + S(y)w[T(x), z] = 0 \text{ for all } x,\ y,\ z,\ w \in R. \tag{4.3.13}$$

Henceforth, using similar approach as we have used after equation (4.3.7) in the proof of Proposition 4.3.1, we get the required result. This finishes the proof of the proposition. □

The main result of the present section is the following theorem which is inspired by Vukman's result [131, Theorem 4].

Theorem 4.3.1. *Let R be a noncommutative 2-torsion free semiprime ring with involution $*$ and $S, T : R \longrightarrow R$ be Jordan left $*$-centralizers. Suppose that*

$$(S(x) \circ T(x))S(x) - S(x)(S(x) \circ T(x)) = 0$$

holds for all $x \in R$. Then $[S(x), T(x)] = 0$ for all $x \in R$. Moreover, if R is a prime ring and $S \neq 0$ $(T \neq 0)$, then there exists $\lambda \in C$ such that $T = \lambda S$ $(S = \lambda T)$.

Proof. By Proposition 4.2.1, we conclude that S and T are reverse left $*$-centralizers. In view of our hypothesis, we have

$$(S(x) \circ T(x))S(x) - S(x)(S(x) \circ T(x)) = 0 \text{ for all } x \in R. \tag{4.3.14}$$

Linearization of relation (4.3.14) yields that

$$\begin{aligned} 0 &= (S(x) \circ T(x))S(y) + (S(x) \circ T(y))S(x) + (S(x) \circ T(y))S(y) \quad (4.3.15) \\ &+ (S(y) \circ T(x))S(x) + (S(y) \circ T(x))S(y) + (S(y) \circ T(y))S(x) \\ &- S(y)(S(x) \circ T(x)) - S(x)(S(x) \circ T(y)) - S(y)(S(x) \circ T(y)) \\ &- S(x)(S(y) \circ T(x)) - S(y)(S(y) \circ T(x)) - S(x)(S(y) \circ T(y)) \end{aligned}$$

for all $x, y \in R$. Replacing x by $-x$ in (4.3.15), we get

$$\begin{aligned} 0 &= (S(x) \circ T(x))S(y) + (S(x) \circ T(y))S(x) - (S(x) \circ T(y))S(y) \quad (4.3.16) \\ &+ (S(y) \circ T(x))S(x) - (S(y) \circ T(x))S(y) - (S(y) \circ T(y))S(x) \\ &- S(y)(S(x) \circ T(x)) - S(x)(S(x) \circ T(y)) + S(y)(S(x) \circ T(y)) \\ &- S(x)(S(y) \circ T(x)) + S(y)(S(y) \circ T(x)) + S(x)(S(y) \circ T(y)) \end{aligned}$$

for all $x, y \in R$. Combining (4.3.15) and (4.3.16), we obtain

$$\begin{aligned} 0 &= 2(S(x) \circ T(x))S(y) + 2(S(x) \circ T(y))S(x) + 2(S(y) \circ T(x))S(x) \\ &- 2S(y)(S(x) \circ T(x)) - 2S(x)(S(x) \circ T(y)) - 2S(x)(S(y) \circ T(x)) \end{aligned}$$

for all $x, y \in R$. Since R is 2-torsion free, the above relation reduces to

$$\begin{aligned} 0 &= (S(x) \circ T(x))S(y) + (S(x) \circ T(y))S(x) + (S(y) \circ T(x))S(x) \\ &- S(y)(S(x) \circ T(x)) - S(x)(S(x) \circ T(y)) - S(x)(S(y) \circ T(x)) \end{aligned} \tag{4.3.17}$$

for all $x, y \in R$. Replacing y by yx in (4.3.17), we obtain

$$\begin{aligned} 0 &= (S(x) \circ T(x))S(x)y^* + (S(x) \circ T(x)y^*)S(x) + (S(x)y^* \circ T(x))S(x) \\ &- S(x)y^*(S(x) \circ T(x)) - S(x)(S(x) \circ T(x)y^*) - S(x)(T(x) \circ S(x)y^*) \end{aligned}$$

for all $x, y \in R$. By using anti-commutator identity, the above relation can be written as

$$\begin{aligned} 0 &= (S(x) \circ T(x))S(x)y^* + (S(x) \circ T(x))y^*S(x) - T(x)[S(x), y^*]S(x) \\ &+ (T(x) \circ S(x))y^*S(x) - S(x)[T(x), y^*]S(x) - S(x)y^*(S(x) \circ T(x)) \\ &- S(x)(S(x) \circ T(x))y^* + S(x)T(x)[S(x), y^*] - S(x)(T(x) \circ S(x))y^* \\ &+ S(x)^2[T(x), y^*] \text{ for all } x, y \in R. \end{aligned} \tag{4.3.18}$$

In view of (4.3.14), the equation (4.3.18) reduces to

$$\begin{aligned} 0 &= (S(x) \circ T(x))y^*S(x) - T(x)[S(x), y^*]S(x) + (T(x) \circ S(x))y^*S(x) \\ &- S(x)[T(x), y^*]S(x) - S(x)y^*(S(x) \circ T(x)) - S(x)(S(x) \circ T(x))y^* \\ &+ S(x)T(x)[S(x), y^*] + S(x)^2[T(x), y^*] \text{ for all } x, y \in R. \end{aligned} \tag{4.3.19}$$

Upon substituting $S(x)^*y$ for y in (4.3.19), we get

$$\begin{aligned} 0 &= (S(x) \circ T(x))y^*S(x)^2 - T(x)[S(x), y^*S(x)]S(x) + (T(x) \circ S(x))y^*S(x)^2 \\ &- S(x)[T(x), y^*S(x)]S(x) - S(x)y^*S(x)(S(x) \circ T(x)) - S(x)(S(x) \circ T(x))y^*S(x) \end{aligned}$$

$$+ \quad S(x)T(x)[S(x), y^*S(x)] + S(x)^2[T(x), y^*S(x)].$$

This implies that

$$\begin{aligned}0 &= (S(x) \circ T(x))y^*S(x)^2 - T(x)[S(x), y^*]S(x)^2 + (T(x) \circ S(x))y^*S(x)^2 \quad (4.3.20)\\ &- S(x)[T(x), y^*]S(x)^2 - S(x)y^*[T(x), S(x)]S(x) - S(x)y^*S(x)(S(x) \circ T(x))\\ &- S(x)(S(x) \circ T(x))y^*S(x) + S(x)T(x)[S(x), y^*]S(x) + S(x)^2[T(x), y^*]S(x)\\ &+ S(x)^2y^*[T(x), S(x)] \text{ for all } x, y \in R.\end{aligned}$$

Application of (4.3.19) yields that

$$S(x)y^*[T(x), S(x)]S(x) - S(x)^2y^*[T(x), S(x)] = 0 \text{ for all } x, y \in R. \tag{4.3.21}$$

Replacing y by $yT(x)^*$ in (4.3.21), we have

$$S(x)T(x)y^*[S(x), T(x)]S(x) - S(x)^2T(x)y^*[S(x), T(x)] = 0 \tag{4.3.22}$$

for all $x, y \in R$. Left multiplying (4.3.21) by $T(x)$ gives

$$T(x)S(x)y^*[S(x), T(x)]S(x) - T(x)S(x)^2y^*[S(x), T(x)] = 0 \tag{4.3.23}$$

for all $x, y \in R$. On combining (4.3.22) and (4.3.23), we obtain

$$[S(x), T(x)]y^*[S(x), T(x)]S(x) - [S(x)^2, T(x)]y^*[S(x), T(x)] = 0 \tag{4.3.24}$$

for all $x, y \in R$. By our hypothesis, we have

$$\begin{aligned}0 &= (S(x) \circ T(x))S(x) - S(x)(S(x) \circ T(x))\\ &= S(x)T(x)S(x) + T(x)S(x)^2 - S(x)^2T(x) - S(x)T(x)S(x)\\ &= T(x)S(x)^2 - S(x)^2T(x) \text{ for all } x \in R.\end{aligned}$$

The above expression can be further written as

$$[S(x)^2, T(x)] = 0 \text{ for all } x \in R. \tag{4.3.25}$$

Using (4.3.25) in (4.3.24), we get

$$[S(x), T(x)]y^*[S(x), T(x)]S(x) = 0 \text{ for all } x, y \in R. \tag{4.3.26}$$

Replacing y by $yS(x)^*$ in (4.3.26), we obtain

$$[S(x), T(x)]S(x)y^*[S(x), T(x)]S(x) = 0 \text{ for all } x, y \in R. \tag{4.3.27}$$

The semiprimeness of R forces that

$$[S(x), T(x)]S(x) = 0 \text{ for all } x \in R. \tag{4.3.28}$$

In view of relation (4.3.25) and (4.3.28), we have

$$S(x)[S(x), T(x)] = 0 \text{ for all } x \in R. \tag{4.3.29}$$

Replacing x by $x + y$ in (4.3.29) and using the same techniques as used to obtain (4.3.17) from (4.3.14), we get

$$S(y)[S(x), T(x)] + S(x)[S(y), T(x)] + S(x)[S(x), T(y)] = 0 \tag{4.3.30}$$

for all $x, y \in R$. Substituting yx for y in (4.3.30), we obtain

$$\begin{aligned} 0 &= S(x)y^*[S(x), T(x)] + S(x)^2[y^*, T(x)] + S(x)[S(x), T(x)]y^* \\ &+ S(x)[S(x), T(x)]y^* + S(x)T(x)[S(x), y^*] \text{ for all } x, y \in R. \end{aligned}$$

This implies

$$S(x)y^*[S(x), T(x)] + S(x)^2[y^*, T(x)] + S(x)T(x)[S(x), y^*] = 0 \tag{4.3.31}$$

for all $x, y \in R$. Thus, we have the relation

$$S(x)y^*[S(x), T(x)] + S(x)^2[y^*, T(x)] + S(x)T(x)[S(x), y^*] = 0 \text{ for all } x, y \in R.$$

Which can be further written in the form

$$S(x)y^*[S(x),T(x)]+S(x)^2y^*T(x)-S(x)T(x)y^*S(x)+S(x)[T(x),S(x)]y^*=0$$

for all $x,y\in R$. Application of (4.3.29) forces that

$$S(x)y^*[S(x),T(x)]+S(x)^2y^*T(x)-S(x)T(x)y^*S(x)=0 \text{ for all } x,y\in R. \quad (4.3.32)$$

Left multiplication of (4.3.32) by $T(x)$ gives

$$T(x)S(x)y^*[S(x),T(x)]+T(x)S(x)^2y^*T(x)-T(x)S(x)T(x)y^*S(x)=0 \quad (4.3.33)$$

for all $x,y\in R$. On substituting $yT(x)^*$ for y in (4.3.32), we have

$$S(x)T(x)y^*[S(x),T(x)]+S(x)^2T(x)y^*T(x)-S(x)T(x)^2y^*S(x)=0 \quad (4.3.34)$$

for all $x,y\in R$. Combining (4.3.33) and (4.3.34), we obtain

$$[S(x),T(x)]y^*[S(x),T(x)]+[S(x)^2,T(x)]y^*T(x)+[T(x),S(x)]T(x)y^*S(x)=0 \quad (4.3.35)$$

for all $x,y\in R$. Using (4.3.25), the above expression reduces to

$$[S(x),T(x)]y^*[S(x),T(x)]+[T(x),S(x)]T(x)y^*S(x)=0 \text{ for all } x,y\in R. \quad (4.3.36)$$

Substituting $zS(x)^*y$ for y in (4.3.36), we get

$$[S(x),T(x)]y^*S(x)z^*[S(x),T(x)]+[T(x),S(x)]T(x)y^*S(x)z^*S(x)=0 \quad (4.3.37)$$

for all $x,y,z\in R$. On the other hand right multiplying to (4.3.36) by $z^*S(x)$, we get

$$[S(x),T(x)]y^*[S(x),T(x)]z^*S(x)+[T(x),S(x)]T(x)y^*S(x)z^*S(x)=0 \quad (4.3.38)$$

for all $x,y,z\in R$. On comparing (4.3.37) and (4.3.38), we obtain

$$[S(x),T(x)]y^*A(x,z)=0 \text{ for all } x,y,z\in R. \quad (4.3.39)$$

Where $A(x,z) = [S(x),T(x)]z^*S(x) - S(x)z^*[S(x),T(x)]$. Substituting $yS(x)^*z$ for y in (4.3.39) gives

$$[S(x),T(x)]z^*S(x)y^*A(x,z) = 0 \text{ for all } x,y,z \in R. \tag{4.3.40}$$

Left multiplying to (4.3.39) by $S(x)z^*$, we get

$$S(x)z^*[S(x),T(x)]y^*A(x,z) = 0 \text{ for all } x,y,z \in R. \tag{4.3.41}$$

From (4.3.40) and (4.3.41), we arrive at $A(x,z)yA(x,z) = 0$ for all $x,y,z \in R$. That is, $A(x,z)RA(x,z) = (0)$ for all $x,z \in R$. The semiprimeness of R forces that $A(x,z) = 0$ for all $x,z \in R$. In other words, we have

$$[S(x),T(x)]z^*S(x) = S(x)z^*[S(x),T(x)] \text{ for all } x,z \in R. \tag{4.3.42}$$

Replacing z by $yT(x)^*$ in (4.3.42), we have

$$[S(x),T(x)]T(x)y^*S(x) = S(x)T(x)y^*[S(x),T(x)] \text{ for all } x,y \in R. \tag{4.3.43}$$

Combining (4.3.36) and (4.3.43), we obtain

$$[S(x),T(x)]y^*[S(x),T(x)] - S(x)T(x)y^*[S(x),T(x)] = 0 \text{ for all } x,y \in R.$$

This further reduces to

$$T(x)S(x)y^*[S(x),T(x)] = 0 \text{ for all } x,y \in R. \tag{4.3.44}$$

If we substitute $yT(x)^*$ for y in (4.3.44), we find that

$$T(x)S(x)T(x)y^*[S(x),T(x)] = 0 \text{ for all } x,y \in R. \tag{4.3.45}$$

Multiplying (4.3.44) from the left side by $T(x)$, we get

$$T(x)^2S(x)y^*[S(x),T(x)] = 0 \text{ for all } x,y \in R. \tag{4.3.46}$$

Subtracting (4.3.46) from (4.3.45), we get

$$T(x)[S(x),T(x)]y^*[S(x),T(x)] = 0 \text{ for all } x,y \in R. \tag{4.3.47}$$

Replacing $T(x)^*y$ for y in (4.3.47), we obtain

$$T(x)[S(x),T(x)]y^*T(x)[S(x),T(x)] = 0 \text{ for all } x,y \in R. \tag{4.3.48}$$

That is,

$$T(x)[S(x),T(x)]RT(x)[S(x),T(x)] = (0) \text{ for all } x \in R.$$

The semiprimeness of R yields that

$$T(x)[S(x),T(x)] = 0 \text{ for all } x \in R. \tag{4.3.49}$$

Replacing y by $T(x)^*y$ in (4.3.43) gives, because of (4.3.49)

$$[S(x),T(x)]y^*T(x)S(x) = 0 \text{ for all } x,y \in R. \tag{4.3.50}$$

Substituting $x+y$ for x in (4.3.28) and using the same approach as we used to obtain (4.3.17) from (4.3.14), we get

$$[S(x),T(x)]S(y) + [S(x),T(y)]S(x) + [S(y),T(x)]S(x) = 0 \tag{4.3.51}$$

for all $x,y \in R$. On substituting yx for y in (4.3.51), we obtain

$$\begin{aligned} 0 &= [S(x),T(x)]S(x)y^* + T(x)[S(x),y^*]S(x) + [S(x),T(x)]y^*S(x) \\ &+ [S(x),T(x)]y^*S(x) + S(x)[y^*,T(x)]S(x) \text{ for all } x,y \in R. \end{aligned}$$

Application of (4.3.28) yields that

$$\begin{aligned} 0 &= [S(x),T(x)]y^*S(x) + T(x)[S(x),y^*]S(x) + [S(x),T(x)]y^*S(x) \\ &+ S(x)[y^*,T(x)]S(x) \text{ for all } x,y \in R. \end{aligned} \tag{4.3.52}$$

This implies that

$$2[S(x),T(x)]y^*S(x)+T(x)[S(x),y^*]S(x)+S(x)[y^*,T(x)]S(x)=0 \qquad (4.3.53)$$

for all $x, y \in R$. This can be further written as

$$\begin{aligned} 0 &= 2[S(x),T(x)]y^*S(x)+T(x)S(x)y^*S(x)-T(x)y^*S(x)^2 \\ &+ S(x)y^*T(x)S(x)-S(x)T(x)y^*S(x) \text{ for all } x,y\in R. \end{aligned}$$

Which reduces to

$$[S(x),T(x)]y^*S(x)+S(x)y^*T(x)S(x)-T(x)y^*S(x)^2=0 \text{ for all } x,y\in R. \qquad (4.3.54)$$

Using (4.3.42) in (4.3.54), we obtain

$$\begin{aligned} 0 &= S(x)y^*[S(x),T(x)]+S(x)y^*T(x)S(x)-T(x)y^*S(x)^2 \\ &= S(x)y^*S(x)T(x)-T(x)y^*S(x)^2 \text{ for all } x,y\in R. \end{aligned}$$

The above expression yields that

$$S(x)y^*S(x)T(x)=T(x)y^*S(x)^2 \text{ for all } x,y\in R. \qquad (4.3.55)$$

Substituting $yT(x)^*$ for y in (4.3.55), we have

$$S(x)T(x)y^*S(x)T(x)=T(x)^2y^*S(x)^2 \text{ for all } x,y\in R. \qquad (4.3.56)$$

Left multiplication to (4.3.55) by $T(x)$ leads to

$$T(x)S(x)y^*S(x)T(x)=T(x)^2y^*S(x)^2 \text{ for all } x,y\in R. \qquad (4.3.57)$$

By combining (4.3.56) and (4.3.57), we arrive at

$$[S(x),T(x)]y^*S(x)T(x)=0 \text{ for all } x,y\in R. \qquad (4.3.58)$$

From (4.3.50) and (4.3.58), we obtain

$$[S(x), T(x)]y^*[S(x), T(x)] = 0 \text{ for all } x, y \in R.$$

That is, $[S(x), T(x)]R[S(x), T(x)] = (0)$. The semiprimeness of R yields that $[S(x), T(x)] = 0$ for all $x \in R$. If R is prime, then in view of Proposition 4.3.1, we get the required result. Thereby the proof of theorem is completed. □

Theorem 4.3.2. *Let R be a noncommutative 2-torsion free semiprime ring with involution $*$ and $S, T : R \longrightarrow R$ be Jordan left $*$-centralizers. Suppose that*

$$[S(x), T(x)]S(x) - S(x)[S(x), T(x)] = 0$$

holds for all $x \in R$. Then $[S(x), T(x)] = 0$ for all $x \in R$. Moreover, if R is a prime ring and $S \neq 0$ $(T \neq 0)$, then there exists $\lambda \in C$ such that $T = \lambda S$ $(S = \lambda T)$.

Proof. We notice that S and T are reverse left $*$-centralizers by Proposition 4.2.1. By the assumption, we have the relation

$$[S(x), T(x)]S(x) - S(x)[S(x), T(x)] = 0 \text{ for all } x \in R. \tag{4.3.59}$$

Replacing x by $x + y$ in (4.3.59), we obtain

$$\begin{aligned} 0 &= [S(x), T(x)]S(y) + [S(x), T(y)]S(x) + [S(x), T(y)]S(y) \\ &+ [S(y), T(x)]S(x) + [S(y), T(x)]S(y) + [S(y), T(y)]S(x) \\ &- S(y)[S(x), T(x)] - S(x)[S(x), T(y)] - S(y)[S(x), T(y)] \\ &- S(x)[S(y), T(x)] - S(y)[S(y), T(x)] - S(x)[S(y), T(y)] \end{aligned}$$

for all $x, y \in R$. Replacing x by $-x$ in (4.3.60), we get

$$\begin{aligned} 0 &= [S(x), T(x)]S(y) + [S(x), T(y)]S(x) - [S(x), T(y)]S(y) \\ &+ [S(y), T(x)]S(x) - [S(y), T(x)]S(y) - [S(y), T(y)]S(x) \\ &- S(y)[S(x), T(x)] - S(x)[S(x), T(y)] + S(y)[S(x), T(y)] \\ &- S(x)[S(y), T(x)] + S(y)[S(y), T(x)] + S(x)[S(y), T(y)] \end{aligned} \tag{4.3.60}$$

for all $x, y \in R$. Combining (4.3.60) and (4.3.60), we obtain

$$\begin{aligned} 0 &= 2[S(x), T(x)]S(y) + 2[S(x), T(y)]S(x) + 2[S(y), T(x)]S(x) \\ &- 2S(y)[S(x), T(x)] - 2S(x)[S(x), T(y)] - 2S(x)[S(y), T(x)] \end{aligned}$$

for all $x, y \in R$. Since R is 2-torsion free, the above relation reduces to

$$\begin{aligned} 0 &= [S(x), T(x)]S(y) + [S(x), T(y)]S(x) + [S(y), T(x)]S(x) \\ &- S(y)[S(x), T(x)] - S(x)[S(x), T(y)] - S(x)[S(y), T(x)] \end{aligned} \tag{4.3.61}$$

for all $x, y \in R$. Substituting yx for y in (4.3.61), we obtain

$$\begin{aligned} 0 &= [S(x), T(x)]S(x)y^* + [S(x), T(x)]y^*S(x) + T(x)[S(x), y^*]S(x) \\ &+ [S(x), T(x)]y^*S(x) + S(x)[y^*, T(x)]S(x) - S(x)y^*[S(x), T(x)] \\ &- S(x)T(x)[S(x), y^*] - S(x)[S(x), T(x)]y^* - S(x)^2[y^*, T(x)] \\ &- S(x)[S(x), T(x)]y^* \text{ for all } x, y \in R. \end{aligned} \tag{4.3.62}$$

Application of (4.3.59) forces that

$$\begin{aligned} 0 &= 2[S(x), T(x)]y^*S(x) + T(x)[S(x), y^*]S(x) + S(x)[y^*, T(x)]S(x) \\ &- S(x)y^*[S(x), T(x)] - S(x)T(x)[S(x), y^*] - S(x)[S(x), T(x)]y^* \\ &- S(r)^2[y^*, T(x)] \text{ for all } x, y \in R. \end{aligned} \tag{4.3.63}$$

Substituting $S(x)^*y$ for y in (4.3.63), we have

$$\begin{aligned} 0 &= 2[S(x), T(x)]y^*S(x)^2 + T(x)[S(x), y^*]S(x)^2 + S(x)[y^*, T(x)]S(x)^2 \\ &+ S(x)y^*[S(x), T(x)]S(x) - S(x)y^*S(x)[S(x), T(x)] - S(x)T(x)[S(x), y^*]S(x) \\ &- S(x)[S(x), T(x)]y^*S(x) - S(x)^2y^*[S(x), T(x)] - S(x)^2[y^*, T(x)]S(x) \end{aligned} \tag{4.3.64}$$

for all $x, y \in R$. Using (4.3.63) in (4.3.64), we conclude that

$$S(x)y^*[S(x), T(x)]S(x) - S(x)^2y^*[S(x), T(x)] = 0 \text{ for all } x, y \in R. \tag{4.3.65}$$

Substituting $yT(x)^*$ for y in the above relation, we obtain

$$S(x)T(x)y^*[S(x),T(x)]S(x) - S(x)^2T(x)y^*[S(x),T(x)] = 0 \qquad (4.3.66)$$

for all $x, y \in R$. On the other hand, left multiplication of (4.3.65) by $T(x)$ gives

$$T(x)S(x)y^*[S(x),T(x)]S(x) - T(x)S(x)^2y^*[S(x),T(x)] = 0 \qquad (4.3.67)$$

for all $x, y \in R$. By comparing (4.3.66) and (4.3.67), we obtain

$$\begin{aligned} 0 &= [S(x),T(x)]y^*[S(x),T(x)]S(x) - [S(x)^2,T(x)]y^*[S(x),T(x)] \\ &= [S(x),T(x)]y^*[S(x),T(x)]S(x) - \\ &\quad ([S(x),T(x)]S(x) + S(x)[S(x),T(x)])y^*[S(x),T(x)] \end{aligned}$$

for all $x, y \in R$. In view of the hypothesis, the above expression reduces to

$$[S(x),T(x)]y^*[S(x),T(x)]S(x) - 2S(x)[S(x),T(x)]y^*[S(x),T(x)] = 0 \qquad (4.3.68)$$

for all $x, y \in R$. If we multiply (4.3.68) by $S(x)$ from left, we get

$$S(x)[S(x),T(x)]y^*[S(x),T(x)]S(x) - 2S(x)^2[S(x),T(x)]y^*[S(x),T(x)] = 0 \qquad (4.3.69)$$

for all $x, y \in R$. On the other hand putting $y[S(x),T(x)]^*$ for y in (4.3.65), we arrive at

$$S(x)[S(x),T(x)]y^*[S(x),T(x)]S(x) - S(x)^2[S(x),T(x)]y^*[S(x),T(x)] = 0 \qquad (4.3.70)$$

for all $x, y \in R$. By combining (4.3.69) and (4.3.70), we obtain

$$S(x)[S(x),T(x)]y^*[S(x),T(x)]S(x) = 0 \text{ for all } x, y \in R.$$

Using (4.3.59) in the above expression, we obtain

$$S(x)[S(x),T(x)]y^*S(x)[S(x),T(x)] = 0 \text{ for all } x, y \in R.$$

Since R is semiprime, it follows that

$$S(x)[S(x),T(x)] = 0 \text{ for all } x \in R. \tag{4.3.71}$$

From (4.3.71) and (4.3.59), we get

$$[S(x),T(x)]S(x) = 0 \text{ for all } x \in R. \tag{4.3.72}$$

The last two expressions are same as the equations (4.3.28) & (4.3.29) and hence, by using similar approach as we used after (4.3.28) & (4.3.29) in the proof of Theorem 4.3.1, we get the required result. The theorem is thereby proved. □

The following results are immediate consequences of the above theorems.

Corollary 4.3.1. *Let R be a noncommutative 2-torsion free semiprime ring with involution $*$ and $T : R \to R$ a Jordan left $*$-centralizer. Suppose $(T(x) \circ x^*)x^* - x^*(T(x) \circ x^*) = 0$ holds for all $x \in R$. Then, T is a reverse $*$-centralizer on R.*

Proof. Taking $S(x) = x^*$ in Theorem 4.3.1 and using the fact that the product '$\circ$' is commutative, we find that

$$[T(x), x^*] = 0 \text{ for all } x \in R.$$

From the above relation, we obtain $T(x^2) = T(x)x^* = x^*T(x)$ for all $x \in R$. This shows that T is Jordan left as well as right $*$-centralizer on R. Hence by Proposition 4.2.1, we conclude that T is a reverse $*$-centralizer on R. □

Similarly, we prove the following:

Corollary 4.3.2. *Let R be a noncommutative 2-torsion free semiprime ring with involution $*$ and $T : R \to R$ a Jordan left $*$-centralizer. Suppose $(T(x) \circ x^*)T(x) - T(x)(T(x) \circ x^*) = 0$ holds for all $x \in R$. Then, T is a reverse $*$-centralizer on R.*

Corollary 4.3.3. *Let R be a noncommutative 2-torsion free semiprime ring with involution $*$ and $T : R \longrightarrow R$ a Jordan left $*$-centralizer. Suppose $[T(x), x^*]x^* - x^*[T(x), x^*] = 0$ holds for all $x \in R$. In this case, T is a reverse $*$-centralizer on R.*

Proof. Substituting $S(x) = x^*$ in Theorem 4.3.2, we obtain

$$[T(x), x^*] = 0 \text{ for all } x \in R.$$

This implies that $T(x^2) = T(x)x^* = x^*T(x)$ for all $x \in R$. In view of Proposition 4.2.1, we conclude that T is a reverse $*$-centralizer on R. $\square$

Corollary 4.3.4. *Let R be a noncommutative 2-torsion free semiprime ring with involution $*$ and $T : R \longrightarrow R$ a Jordan left $*$-centralizer. Suppose $[T(x), x^*]T(x) - T(x)[T(x), x^*] = 0$ holds for all $x \in R$. In this case, T is a reverse $*$-centralizer on R.*

4.4 Applications

In this section, we present some applications of our previous results. Following [2], an additive mapping $F : R \longrightarrow R$ is called a generalized $*$-derivation (resp. generalized reverse $*$-derivation) if there exists a $*$- derivation (resp. reverse $*$- derivation) d such that $F(xy) = F(x)y^* + xd(y)$ (resp. $F(xy) = F(y)x^* + yd(x)$) holds for all $x, y \in R$. For some fixed elements a and b of R, an additive function $F_{a,b} : R \longrightarrow R$ is called a generalized Jordan inner $*$-derivation if $F_{a,b}(x) = ax^* + xb$ for all $x \in R$. It is straightforward to note that if $F_{a,b}$ is a generalized Jordan inner $*$-derivation, then for any $x \in R$, we have

$$\begin{aligned} F_{a,b}(x^2) &= F_{a,b}(x)x^* + x(xb - bx^*) \\ &= F_{a,b}(x)x^* + xI_{-b}(x) \end{aligned}$$

where I_{-b} is a Jordan inner $*$-derivation. In view of the above observation, the concept of generalized Jordan $*$-derivation is defined as follows: an additive mapping $F : R \longrightarrow R$ is called a generalized Jordan $*$-derivation with an associated Jordan $*$- derivation d if $F(x^2) = F(x)x^* + xd(x)$ holds for all $x \in R$. An additive mapping $F : R \longrightarrow R$ is called a generalized Jordan triple $*$-derivation if there exists a Jordan triple $*$-derivation $d : R \longrightarrow R$ such that $F(xyx) = F(x)y^*x^* + xd(y)x^* + xyd(x)$ holds for all $x, y \in R$. Any generalized Jordan $*$-derivation on a 2-torsion free $*$-ring is a generalized Jordan triple $*$-derivation (see for example [9, Lemma 2.4]) but the converse need not be true in general. In [61, Theorem 5.4], Fošner and Iliševic proved that any generalized Jordan triple $*$-derivation on a 2-torsion free semiprime $*$-ring is a generalized Jordan

$*$-derivation. One may observe that the concept of generalized Jordan $*$-derivations includes the concepts of Jordan $*$-derivations and Jordan left $*$-centralizers when $d = 0$. Hence, it would be interesting to extend some results concerning these notions to generalized Jordan $*$-derivations. Moreover, it is easy to see that F is a generalized Jordan $*$-derivation of R if and only if F is of the form $F = d + T$, where d is a Jordan $*$-derivation and T is a Jordan left $*$-centralizer of R. Thus, we can write $T = F - d$. In the proof of our next theorem, we are using this technique which can be regarded as a contribution to the theory of $*$-centralizers in rings with involution. Recently, Brešar and Vukman [45] proved that a noncommutative prime $*$-ring of characteristic different from 2 is normal if and only if there exists a nonzero Jordan $*$-derivation d such that $[d(x), x] = 0$ for all $x \in R$. In view of above result, it is natural to ask that: What can we say about the normality of R if Jordan $*$-derivation d is replaced by generalized Jordan $*$-derivation F of R ? In this section, we have succeeded in establishing the following result for prime rings with involution.

Theorem 4.4.1. *Let R be a noncommutative prime ring with involution $*$ of characteristic different from 2. Then the following conditions are mutually equivalent:*

(i) *R is normal*

(ii) *there exists a nonzero commuting generalized Jordan $*$-derivation F with an associated Jordan $*$-derivation d, which is also commuting on R.*

Proof. Suppose (i) holds, then the mapping $x \mapsto x^* + x$ is a nonzero commuting generalized Jordan $*$-derivation with an associated commuting Jordan $*$-derivation $d(x) = x - x^*$ for all $x \in R$. On the other hand, assume that (ii) holds. Then we can always write $T = F + d$ or $T = F - d$. This gives

$$\begin{aligned} T(x^2) &= F(x^2) - d(x^2) \\ &= F(x)x^* + xd(x) - d(x)x^* - xd(x) \\ &= (F(x) - d(x))x^* \\ &= T(x)x^* \text{ for all } x \in R. \end{aligned}$$

That is, $T(x^2) = T(x)x^*$ for all $x \in R$ and hence T is Jordan left $*$-centralizer on R.

Moreover, in view of our hypothesis, we are forced to conclude that

$$\begin{aligned}[T(x), x] &= [F(x) - d(x), x] \\ &= [F(x), x] - [d(x), x] \\ &= 0 \text{ for all } x \in R.\end{aligned}$$

Thus we are able to find a Jordan left $*$-centralizer on R which is commuting and nonzero on R. Hence, Theorem 4.2.2 yields the required result. Thereby the proof is completed. $\square$

The above theorem have the following two interesting consequences:

Corollary 4.4.1. *[45, Theorem 3] Let R be a noncommutative prime ring with involution $*$ of characteristic different from 2. Then the following conditions are mutually equivalent:*

(i) R is normal

(ii) there exists a nonzero commuting Jordan $*$-derivation d on R.

Corollary 4.4.2. *Let R be a noncommutative prime ring with involution $*$ of characteristic different from 2. Then the following conditions are mutually equivalent:*

(i) R is normal

(ii) there exists a nonzero commuting generalized Jordan triple $*$-derivation F with an associated nonzero Jordan triple $*$-derivation d which is also commuting on R.

Proof. In view of Lemma 1.3.7, on a 2-torsion free prime ring with involution, every generalized Jordan triple $*$- derivation is a generalized Jordan $*$-derivation and every Jordan triple $*$-derivation is a Jordan $*$-derivation. Hence, the proof follows from Theorem 4.4.1. $\square$

CHAPTER-5

On Jordan *-Mappings in Rings with Involution and their Applications

Chapter 5

On Jordan $*$-Mappings in Rings with Involution and their Applications

5.1 Introduction

In the year 1957, Herstein [71] introduced the notion of a function which he called Jordan derivation and defined as: an additive mapping $d : R \to R$ is said to be a Jordan derivation of R if $d(x^2) = d(x)x + xd(x)$ holds for all $x \in R$. Every derivation is obviously a Jordan derivation but the converse need not be true in general (see [24, Example 3.2.1]). A classical result due to Herstein [71] asserts that any Jordan derivation on a prime ring of characteristic different from two is a derivation. A brief proof of Herstein's theorem can be found in [44]. Cusack [53] generalized Herstein's result to 2-torsion free semiprime rings (see [38] for an alternative proof). This famous result was further generalized by many authors in various directions (viz.; [20, 26, 27, 39, 47, 61, 135], where further references can be found).

Let R be a ring with involution $*$. According to Brešar and Vukman [45], an additive mapping $d : R \to R$ is called a $*$-derivation (resp. Jordan $*$-derivation) if $d(xy) = d(x)y^* + xd(y)$ (resp. $d(x^2) = d(x)x^* + xd(x)$) holds for all $x, y \in R$. Note that the mapping $x \mapsto ax^* - xa$, where a is a fixed element in R, is a Jordan $*$-derivation; such Jordan $*$-derivations are said to be inner. One might expect that any Jordan $*$-derivation on a 2-torsion free semiprime $*$-ring is a $*$-derivation, but this is not the case.

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It is easy to prove that there exist no nonzero $*$-derivations on noncommutative prime $*$-rings (see [45] for details). The study of Jordan $*$-derivations has been motivated by the problem of the representativity of quadratic forms by bilinear forms (for results concerning this problem we refer to [87, 116, 117, 118, 123, 126] and references there in). It turns out that the question whether each quadratic form can be represented by some bilinear form is intimately connected with the question whether every Jordan $*$-derivation is inner, as shown by Šemrl [117]. In [45], Brešar and Vukman studied some algebraic properties of Jordan $*$-derivations. As a special case of [45, Theorem 1], they showed that every Jordan $*$-derivation of a complex algebra A with unit element is inner. Further, Brešar and Zalar [49] proved that Jordan $*$-derivations of a rather wide class of complex $*$-algebras (in general without unit) can be represented by double centralizers (viz.; [49, Theorem 2.1]). As an application, they obtained the structure of Jordan $*$-derivations on the algebra of all compact linear operators on a complex Hilbert space in [49]. Some related papers on this subject can be found in [9, 61, 80, 135, 147] etcetera.

Section 5.2 is devoted to the study of $*$-derivations and Jordan $*$-derivations in prime and semiprime rings with involution, and it is shown that on 2-torsion free semiprime ring R with involution, an additive mapping $d : R \to R$ satisfying the relation $d(xyx) = d(xy)x^* + xyd(x)$ for all $x, y \in R$ is a $*$-derivation. Moreover, following result concerning Jordan $*$-derivations is obtained: Let R be a prime ring with involution $*$, of characteristic different from 2. Let d be a nonzero Jordan $*$-derivation of R such that $[d(x), x] = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$. Then R is commutative. Further, motivated by the definition of $*$-derivation and Jordan $*$-derivation, we introduce the notion of left $*$-derivation and Jordan left $*$-derivation and obtain similar results in the setting of these mappings.

In Section 5.3, we not only introduce the notion of symmetric Jordan $*$-biderivation and symmetric Jordan triple $*$-biderivation, but also establish a set of conditions under which the two concepts are equivalent.

The final section of this chapter is devoted to study the applications of our results obtained in previous section. In fact, besides proving some other results, it is shown that in a $(m+n)!$-torsion free noncommutative prime ring with involution $*$ having an identity element e, which admits Jordan $*$-derivations d, g such that $d(x^m)x^n \pm x^n g(x^m) = 0$ for all $x \in R$, where m, n are non-negative integers and $S(R) \cap Z(R) \neq (0)$, implies $d = g = 0$.

5.2 Jordan $*$-derivations in rings

Let R be a ring with involution $*$. An additive mapping $d : R \to R$ is said to be a $*$-derivation (resp. Jordan $*$-derivation) if $d(xy) = d(x)y^* + xd(y)$ (resp. $d(x^2) = d(x)x^* + xd(x)$) holds for all $x, y \in R$. The concept of Jordan $*$-derivation appears for the first time in the work of Brešar and Vukman [45]. The notion of Jordan $*$-derivations arise naturally in the theory of representability of quadratic functionals with sesquilinear functions (see for example [117] and [118]). For results concerning this theory we refer to ([124], [126] and [147], and references there in). Following [135], an additive mapping $d : R \to R$ is called a Jordan triple $*$-derivation if $d(xyx) = d(x)y^*x^* + xd(y)x^* + xyd(x)$ holds for all $x, y \in R$. One can easily prove that every Jordan $*$-derivation on a 2-torsion free $*$-ring is a Jordan triple $*$-derivation of R. However, the converse of this statement need not be true in general. In [135], Vukman showed that the converse holds if R is 6-torsion free semiprime $*$-ring. Recently, Fošner and Iliševic [61] generalized above mentioned result as follows:

Theorem 5.2.1. *Let R be a 2-torsion free semiprime ring with involution $*$ and let $d : R \longrightarrow R$ be an additive mapping satisfying the relation*

$$d(xyx) = d(x)y^*x^* + xd(y)x^* + xyd(x)$$

for all $x, y \in R$. In this case, d is a Jordan $*$-derivation.

Motivated by the above theorem, we prove the following result:

Theorem 5.2.2. *Let R be a 2-torsion free semiprime ring with involution $*$ and let $d : R \to R$ be an additive mapping . Suppose that*

$$d(xyx) = d(xy)x^* + xyd(x)$$

holds for all pairs $x, y \in R$. In this case d is a $*$-derivation.

For developing the proof of Theorem 5.2.2, we need the following lemma:

Lemma 5.2.1. *Let R be a semiprime ring with involution $*$, and let $f : R \to R$ be an additive mapping. Suppose that either*

$$f(x)x^* = 0 \text{ or } x^*f(x) = 0$$

holds for all $x \in R$. In both the cases $f = 0$.

Proof. We first consider the case $f(x)x^* = 0$ for all $x \in R$. Linearizing this relation, we have

$$f(x)y^* + f(y)x^* = 0 \text{ for all } x, y \in R. \tag{5.2.1}$$

Replacing y by y^2 in (5.2.1), we have

$$f(x)(y^*)^2 + f(y^2)x^* = 0 \text{ for all } x, y \in R. \tag{5.2.2}$$

Right multiplying (5.2.1) by y^*, we obtain

$$f(x)(y^*)^2 + f(y)x^*y^* = 0 \text{ for all } x, y \in R. \tag{5.2.3}$$

Combining (5.2.2) and (5.2.3), we get

$$f(y^2)x^* - f(y)x^*y^* = 0 \text{ for all } x, y \in R. \tag{5.2.4}$$

Substituting $f(y)^*x$ for x in (5.2.4), we find that

$$f(y^2)x^*f(y) - f(y)x^*f(y)y^* = 0 \text{ for all } x, y \in R.$$

In view of our hypothesis, we conclude that

$$f(y^2)x^*f(y) = 0 \text{ for all } x, y \in R. \tag{5.2.5}$$

Right multiplying (5.2.4) by $f(y)$ and combining with (5.2.5), we arrive at

$$f(y)x^*y^*f(y) = 0 \text{ for all } x, y \in R.$$

This further implies that $y^*f(y)x^*y^*f(y) = 0$ for all $x, y \in R$. The semiprimeness of R yields that

$$x^*f(x) = 0 \text{ for all } x \in R. \tag{5.2.6}$$

Right multiplying (5.2.1) by $f(x)$ gives because of (5.2.6) $f(x)y^*f(x) = 0$ for all $x, y \in R$. Thus by the semiprimeness of R, we conclude that $f(x) = 0$ for all $x \in R$.

By the same argument, we obtain the similar conclusion in the case $x^*f(x) = 0$ for all $x \in R$. This proves the lemma completely. □

Proof of Theorem 5.2.2. By the hypothesis, we have

$$d(xyx) = d(xy)x^* + xyd(x) \text{ for all } x, y \in R. \tag{5.2.7}$$

Linearizing (5.2.7), we get

$$d(xyz + zyx) = d(xy)z^* + d(zy)x^* + xyd(z) + zyd(x) \text{ for all } x, y, z \in R. \tag{5.2.8}$$

Replacing z by x^2, the above relation gives

$$d(xyx^2 + x^2yx) = d(xy)(x^*)^2 + d(x^2y)x^* + xyd(x^2) + x^2yd(x) \tag{5.2.9}$$

for all $x, y \in R$. Substituting $xy + yx$ for y in (5.2.7), we get

$$\begin{aligned} d(xyx^2 + x^2yx) &= d(x^2y)x^* + d(xy)(x^*)^2 + xyd(x)x^* \\ &+ x^2yd(x) + xyxd(x) \end{aligned} \tag{5.2.10}$$

for all $x, y \in R$. Comparing (5.2.9) and (5.2.10), we obtain

$$xyA(x) = 0 \text{ for all } x, y \in R, \tag{5.2.11}$$

where $A(x) = d(x^2) - d(x)x^* - xd(x)$ for all $x \in R$. Right multiplying (5.2.11) by x and left multiplying by $A(x)$, we get $A(x)xyA(x)x = 0$ for all $x, y \in R$. In view of the semiprimeness of R, we are forced to conclude that

$$A(x)x = 0 \text{ for all } x \in R. \tag{5.2.12}$$

Substituting $A(x)yx$ for y in (5.2.11), we obtain $xA(x)yxA(x) = 0$ for all $x, y \in R$. The last expression forces that

$$xA(x) = 0 \text{ for all } x \in R. \tag{5.2.13}$$

Linearizing (5.2.12), we get

$$B(x,y)x + A(x)y + B(x,y)y + A(y)x = 0 \text{ for all } x, y \in R,$$

where $B(x,y) = d(xy + yx) - d(x)y^* - d(y)x^* - xd(y) - yd(x)$. Putting in the above relation $-x$ for x, we have

$$B(x,y)x + A(x)y - B(x,y)y - A(y)x = 0 \text{ for all } x, y \in R.$$

Comparing the above two equations and using the 2-torsion freeness of R, we obtain

$$B(x,y)x + A(x)y = 0 \text{ for all } x, y \in R.$$

Right multiplication of the above equation by $A(x)$ gives because of (5.2.13) $A(x)yA(x) = 0$ for all $x, y \in R$. Whence it follows $A(x) = 0$ for all $x \in R$. That is, $d(x^2) = d(x)x^* + xd(x)$ for all $x \in R$. This shows d is a Jordan $*$-derivation and hence in view of Lemma 1.3.7, we have

$$d(xyx) = d(x)y^*x^* + xd(y)x^* + xyd(x) \text{ for all } x, y \in R. \tag{5.2.14}$$

Combining (5.2.7) and (5.2.14), we obtain

$$(d(xy) - d(x)y^* - xd(y))x^* = 0 \text{ for all } x, y \in R.$$

For any fixed element y of R we have an additive mapping $x \mapsto d(xy) - d(x)y^* - xd(y)$. Application of Lemma 5.2.1 yields that $d(xy) - d(x)y^* - xd(y) = 0$ for all $x, y \in R$. That is, d is a $*$-derivation. Thereby the proof of the theorem is completed. $\square$

We now prove another result in the spirit of Theorem 5.2.2.

Theorem 5.2.3. *Let R be a 2-torsion free semiprime ring with involution $*$ and let $d : R \to R$ be an additive mapping. Suppose that*

$$d(xyx) = d(x)y^*x^* + xd(yx)$$

holds for all pairs $x, y \in R$. In this case d is a $*$-derivation.

Proof. Using the similar approach as we have used in the proof of Theorem 5.2.2 and using Lemma 1.3.15 instead of Lemma 5.2.1, we get the required result. $\square$

In [45, Theorem 3], Brešar and Vukman established the following result:

Theorem 5.2.1. *Let R be a noncommutative prime ring with involution $*$ such that $char(R) \neq 2$. Then R is normal if and only if there exists a nonzero Jordan $*$-derivation $d : R \to R$ such that $[d(x), x] = 0$ for all $x \in R$.*

The above result motivated us to prove the next theorem:

Theorem 5.2.4. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. Let d be a nonzero Jordan $*$-derivation of R such that $[d(x), x] = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$. Then, R is commutative.*

We first prove the following key lemma.

Lemma 5.2.2. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If $S(R) \subseteq Z(R)$, then R is commutative.*

Proof. By the assumption, we have $S(R) \subseteq Z(R)$. This gives $[k, x] = 0$ for all $k \in S(R)$ and $x \in R$. If $h \in H(R)$, $k \in S(R)$, then $hk + kh \in S(R)$ and hence $0 = [hk + kh, x] = [h, x]k + k[h, x] = 2k[h, x]$ for all $x \in R$ and $h \in H(R)$, $k \in S(R)$. Since $\text{char}(R) \neq 2$, we get $k[h, x] = 0$ for all $x \in R$ and $h \in H(R)$, $k \in S(R)$. Using the primeness of R we get either $S(R) = (0)$ or $H(R) \subseteq Z(R)$. If $S(R) = (0)$, then for every $x \in R$, $2x \in H(R)$. Therefore, we obtain $(2x2y)^* = (2y)^*(2x)^*$ for all $x, y \in R$. This implies that $2x2y = 2y2x$. That is, $4xy = 4yx$ for all $x, y \in R$. Since $\text{char}(R) \neq 2$, we get $xy = yx$ for all $x, y \in R$, which proves that R is commutative. On the other hand, suppose $H(R) \subseteq Z(R)$. Since R is a prime ring with involution, of characteristic different from 2, every $x \in R$ can be represented as $2x = h + k$, where $h \in H(R)$ and $k \in S(R)$. This gives $2R \subseteq Z(R)$. Since $\text{char}(R) \neq 2$, we get $R \subseteq Z(R)$. Hence, R is commutative. This proves the lemma. $\square$

Proof of Theorem 5.2.4. We are given that $d : R \to R$ is a Jordan $*$-derivation such that $[d(x), x] = 0$ for all $x \in R$. Hence in view of Lemma 1.3.5, we conclude that $d(x) = px + \lambda(x)$ for all $x \in R$, where $p \in C$ and $\lambda : R \to C$. Suppose $p \neq 0$. Since d is a nonzero Jordan $*$-derivation. So the above relation yields that $px^2 + \lambda(x^2) = pxx^* + px^2 + \lambda(x)x^* + \lambda(x)x$. This implies that

$$0 = pxx^* + \lambda(x)x^* + \lambda(x)x - \lambda(x^2) \text{ for all } x \in R. \tag{5.2.15}$$

If $x = k \in S(R)$, skew symmetric element, then we arrive at $0 = pk^2 + \lambda(k^2)$. Therefore $[pk^2, y] = 0$ for all $y \in R$ and $k \in S(R)$. It is easy to verify that $p[k^2, y] = 0$ for all $y \in R$ and $k \in S(R)$. In case $p \neq 0$, we get $k^2 \in Z(R)$ for all $x \in S(R)$. On the other hand, if $p = 0$, then $d(x) = \lambda(x)$ and hence $[d(x), y] = 0$ for all $x, y \in R$. Replacing x by x^2 and using the fact d is a Jordan $*$-derivation, we get $0 = [d(x^2), y] = [d(x)x^* + xd(x), y] = d(x)[x^*, y] + [x, y]d(x)$. This further implies that $d(x)[x + x^*, y] = 0$ for all $x, y \in R$. Replacing y by yz in the last expression, we obtain $d(x)y[x+x^*, z] = 0$ for all $x, y, z \in R$. Thus for each $x \in R$, by the primeness of R either $d(x) = 0$ or $[x + x^*, z] = 0$. Now let $A = \{x \in R | d(x) = 0\}$ and $B = \{x \in R | \ [x + x^*, z] = 0 \text{ for all } z \in R\}$. Thus A and B are additive subgroups of R and $R = A \cup B$. But a group can not be a union of two of its proper subgroups and hence either $R = A$ or $R = B$. Since we have assumed $d \neq 0$, we have $R = B$ that is, $[x + x^*, z] = 0$ for all $x, z \in R$. Replacing x by $h + k$, where $h \in H(R)$ and $k \in S(R)$, we get $2[h, z] = 0$. Since $char(R) \neq 2$, we obtain $[h, z] = 0$ for all $h \in H(R)$ and $z \in R$. That is, $h \in Z(R)$ for all $h \in H(R)$. This further implies that $k^2 \in Z(R)$ for all $k \in S(R)$. Thus in both cases $k^2 \in Z(R)$ for all $k \in S(R)$. Now since $S(R) \cap Z(R) \neq (0)$. Let $0 \neq k_0 \in S(R) \cap Z(R)$ and let k be an arbitrary element of $S(R)$. Then k^2, k_0^2 and $(k + k_0)^2 = k^2 + k_0^2 + 2kk_0$ are all in $Z(R)$; and it follows that $2kk_0 \in Z(R)$ and hence $k \in Z(R)$ for all $k \in S(R)$. This implies that R is commutative in view of Lemma 5.2.2. Thereby completing the proof of the theorem. □

Inspired by the definition of $*$-derivation (resp. Jordan $*$-derivation), we define left $*$-derivation (resp. Jordan left $*$-derivation) as follows:

Definition 5.2.1. *Let R be a ring with involution $*$. An additive mapping d of R into itself is called a left $*$-derivation (resp. Jordan left $*$-derivation) of R if $d(xy) = y^*d(x) + xd(y)$ (resp. $d(x^2) = x^*d(x) + xd(x)$) holds for all $x, y \in R$.*

It is to remark that, in case of a commutative ring with involution, any Jordan $*$-derivation is a Jordan left $*$-derivation and vice versa. However, the above statement need not be true for arbitrary rings. The following example justifies this fact:

Example 5.2.1. *Let* $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \,\middle|\, a, b, c \in \mathbb{Z} \right\}$. *Then, R is a noncommutative ring under usual matrix operations. Define the maps $d : R \longrightarrow R$, and $* : R \longrightarrow R$ as follows* $d\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & c & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}$.

Then, it is easy to verify that d is a Jordan left $*$-derivation on R. However, d is not a Jordan $*$-derivation on R.

Proposition 5.2.1. *Let R be a noncommutative prime ring with involution $*$ and let $d : R \to R$ be a left $*$-derivation on R. Then, $d = 0$.*

Proof. By the hypothesis, we have $d(xyz) = d(x(yz)) = z^*y^*d(x) + xz^*d(y) + xyd(z)$ for all $x, y, z \in R$. On the other hand, we have $d(xyz) = d((xy)z) = z^*y^*d(x) + z^*xd(y) + xyd(z)$ for all $x, y, z \in R$. Comparing the above two expressions, we get $[x, z^*]d(y) = 0$ for all $x, y, z \in R$. Replacing x by xt in the last relation, we obtain $0 = [xt, z^*]d(y) = [x, z^*]td(y) + x[t, z^*]d(y) = [x, z^*]td(y)$ for all $x, y, z, t \in R$. This implies that $[x, z^*]Rd(y) = (0)$ for all $x, y, z \in R$. The primeness of R yields that either $[x, z^*] = 0$ for all $x, z \in R$, or $d(y) = 0$ for all $y \in R$. Since R is noncommutative, we are forced to conclude that $d(y) = 0$ for all $y \in R$. Hence $d = 0$. This proves the proposition. □

The next result is in spirit of Theorem 5.2.4 above in the setting of Jordan left $*$-derivations.

Theorem 5.2.5. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. Let d be a nonzero Jordan left $*$-derivation of R such that $[d(x), x] = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$. Then, R is commutative.*

Proof. We are given that $d : R \to R$ is a Jordan left $*$-derivation such that $[d(x), x] = 0$ for all $x \in R$. Hence in view of Lemma 1.3.5, we conclude that $d(x) = px + \lambda(x)$ for all $x \in R$, where $p \in C$ and $\lambda : R \to C$. Suppose $p \neq 0$. Since d is a nonzero Jordan left $*$-derivation, so the above relation yields that

$$px^2 + \lambda(x^2) = px^*x + px^2 + x^*\lambda(x) + x\lambda(x) \text{ for all } x \in R.$$

This implies that

$$px^*x + x^*\lambda(x) + x\lambda(x) - \lambda(x^2) = 0 \text{ for all } x \in R.$$

This equation is same as (5.2.15). Henceforth, using the same technique with necessary variations as we have used in the proof of Theorem 5.2.4, we get the required result. This completes the proof of the theorem. □

5.3 Jordan $*$-biderivations in semiprime rings

A symmetric biadditive map $B : R \times R \to R$ is called a symmetric biderivation if $B(xy, z) = B(x, z)y + xB(y, z)$ is fulfilled for all $x, y, z \in R$. The concept of a symmetric biderivation was introduced by Maksa in [101] (see also [102], where an example can be found). A symmetric biadditive map $B : R \times R \to R$ is said to be a symmetric Jordan biderivation if $B(x^2, z) = B(x, z)x + xB(x, z)$ holds for all $x, z \in R$. Following [6], a symmetric biadditive map $B : R \times R \to R$ is called a symmetric $*$-biderivation if $B(xy, z) = B(x, z)y^* + xB(y, z)$ holds for all $x, y, z \in R$, where R is a ring with involution $*$. Motivated by the definition of Jordan $*$-derivation and Jordan triple $*$-derivation in rings with involution, we introduce the concept of symmetric Jordan $*$-biderivation and symmetric Jordan triple $*$-biderivation as follows: a symmetric biadditive map $D : R \times R \to R$ is said to be a symmetric Jordan $*$-biderivation if $D(x^2, z) = D(x, z)x^* + xD(x, z)$ holds for all $x, z \in R$. A symmetric biadditive map $D : R \times R \to R$ is called a symmetric Jordan triple $*$-biderivation if $D(xyx, z) = D(x, z)y^*x^* + xD(y, z)x^* + xyD(x, z)$ holds for all $x, y, z \in R$. It is obvious to see that every symmetric Jordan $*$-biderivation on a 2-torsion free ring with involution is a symmetric Jordan triple $*$-biderivation. But the converse need not be true in general. In the present section, our aim is to establish a set of conditions under which every symmetric Jordan triple $*$-biderivation on a ring with involution $*$ is a symmetric Jordan $*$-biderivation. More precisely, we prove that on a 2-torsion free semiprime ring with involution, every symmetric Jordan triple $*$-biderivation is a symmetric Jordan $*$-biderivation.

We begin our discussion with the following lemma whose proof can be found at the beginning of [80, Section 2].

Lemma 5.3.1. *Let R is a semiprime ring with involution $*$, if $x \in R$, then $yxy^* = 0$ for all $y \in R$ implies that $x = 0$.*

Lemma 5.3.2. *Let R be a 2-torsion free ring with involution $*$. If $D : R \times R \to R$ is a symmetric Jordan $*$-biderivation, then the following hold:*

(*i*) $D(xy + yx, z) = D(x, z)y^* + D(y, z)x^* + xD(y, z) + yD(x, z)$ *for all* $x, y, z \in R$;

(*ii*) $D(xyx, z) = D(x, z)y^*x^* + xD(y, z)x^* + xyD(x, z)$ *for all* $x, y, z \in R$;

(*iii*) $D(xyt + tyx, z) = D(x, z)y^*t^* + xD(y, z)t^* + xyD(t, z)$
$+ D(t, z)y^*x^* + tD(y, z)x^* + tyD(x, z)$ *for all* $x, y, z, t \in R$.

Proof. (*i*) For any $x, y, z \in R$, we have

$$\begin{aligned} & D(xy + yx, z) && (5.3.1)\\ = \;& D((x + y)^2, z) - D(x^2, z) - D(y^2, z) \\ = \;& D(x + y, z)(x + y)^* + (x + y)D(x + y, z) - D(x, z)x^* \\ - \;& xD(x, z) - D(y, z)y^* - yD(y, z) \\ = \;& D(x, z)y^* + D(y, z)x^* + xD(y, z) + yD(x, z). \end{aligned}$$

(*ii*) Replacing y by $xy + yx$ in (*i*), we get

$$\begin{aligned} & D(x(xy + yx) + (xy + yx)x, z) && (5.3.2)\\ = \;& D(x, z)(xy + yx)^* + D(xy + yx)x^* + xD(xy + yx, z) + (xy + yx)D(x, z) \\ = \;& D(x, z)y^*x^* + D(x, z)x^*y^* + D(x, z)y^*x^* + D(y, z)(x^*)^2 \\ + \;& xD(y, z)x^* + yD(x, z)x^* + xD(x, z)y^* + xD(y, z)x^* + x^2D(y, z) \\ + \;& xyD(x, z) + xyD(x, z) + yxD(x, z) \text{ for all } x, y, z \in R. \end{aligned}$$

On the other hand, we have

$$D(x(xy + yx) + (xy + yx)x, z) \qquad (5.3.3)$$

$$
\begin{aligned}
&= D(x^2y + yx^2, z) + 2D(xyx, z) \\
&= D(x,z)x^*y^* + xD(x,z)y^* + D(y,z)(x^*)^2 \\
&+ x^2D(y,z) + yD(x,z)x^* + yxD(x,z) \\
&+ 2D(xyx, z) \text{ for all } x, y, z \in R.
\end{aligned}
$$

Comparing (5.3.2) and (5.3.3), we get

$$2D(xyx, z) = 2D(x,z)y^*x^* + 2xD(y,z)x^* + 2xyD(x,z) \quad \text{for all } x, y, z \in R.$$

Since R is 2-torsion free, the last expression yields the required result.

(iii) Putting $x + t$ instead of x in (ii), we get

$$
\begin{aligned}
& D((x+t)y(x+t), z) \\
&= D(x+t, z)y^*(x^* + t^*) + (x+t)D(y,z)(x^* + t^*) \\
&+ (x+t)yD(x+t, z) \\
&= D(x,z)y^*x^* + D(x,z)y^*t^* + D(t,z)y^*x^* + D(t,z)y^*t^* \\
&+ xD(y,z)x^* + xD(y,z)t^* + tD(y,z)x^* + tD(y,z)t^* \\
&+ xyD(x,z) + xyD(t,z) + tyD(x,z) + tyD(t,z) \text{ for all } x, y, z, t \in R.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
& D((x+t)y(x+t), z) \\
&= D(xyx, z) + D(tyt, z) + D(xyt + tyx, z) \\
&= D(x,z)y^*x^* + xD(y,z)x^* + xyD(x,z) \\
&+ D(t,z)y^*t^* + tD(y,z)t^* + tyD(t,z) \\
&+ D(xyt + tyx, z) \text{ for all } x, y, z, t \in R.
\end{aligned}
$$

Comparing so obtained relations, we get the desired result. □

We are now ready to prove the main result of this section:

Theorem 5.3.1. *Let R be a 2-torsion free semiprime ring with involution $*$. Then* *every symmetric Jordan triple $*$-biderivation $D : R \times R \to R$ is a symmetric Jordan $*$-biderivation.*

Proof. By the assumption, we have

$$D(xyx, z) = D(x, z)y^*x^* + xD(y, z)x^* + xyD(x, z) \text{ for all } x, y, z \in R. \tag{5.3.4}$$

In view of Lemma 5.3.2 (iii), we have

$$\begin{aligned} D(xyt + tyx, z) &= D(x, z)y^*t^* + xD(y, z)t^* + xyD(t, z) \\ &+ D(t, z)y^*x^* + tD(y, z)x^* + tyD(x, z) \text{ for all } x, y, z, t \in R. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} D((xy)^2, z) &= D(xyxy, z) = D(xy(xy) + (xy)yx - xy^2x, z) \\ &= D(xy(xy) + (xy)yx, z) - D(xy^2x, z) \\ &= D(x, z)(y^*)^2x^* + xD(y, z)y^*x^* + xyD(xy, z) \\ &+ D(xy, z)y^*x^* + xyD(y, z)x^* + xy^2D(x, z) \\ &- D(x, z)(y^*)^2x^* - xD(y^2, z)x^* - xy^2D(x, z) \text{ for all } x, y, z \in R. \end{aligned}$$

It follows that

$$\begin{aligned} & D((xy)^2, z) - D(xy, z)y^*x^* - xyD(xy, z) \\ + \; & x(D(y^2, z) - D(y, z)y^* - yD(y, z)x^*) = 0 \text{ for all } x, y, z \in R. \end{aligned} \tag{5.3.5}$$

Therefore, relation (5.3.5) can be written as

$$\Delta(xy) + x\Delta(y)x^* = 0 \text{ for all } x, y \in R. \tag{5.3.6}$$

where

$$\Delta(x) = D(x^2, z) - D(x, z)x^* - xD(x, z) \text{ for all } x, z \in R.$$

In view of relation (5.3.6), we find that

$$2ty\Delta(x)y^*t^* = ty\Delta(x)y^*t^* + ty\Delta(x)y^*t^*$$

$$
\begin{aligned}
&= -t\Delta(yx)t^* - \Delta((ty)x) \\
&= -t\Delta(yx)t^* - \Delta(tyx) \\
&= \Delta(tyx) - \Delta(tyx) \\
&= 0 \text{ for all } x, y, t \in R.
\end{aligned}
$$

Thus $2ty\Delta(x)y^*t^* = 0$ for all $x, y, t \in R$. Since R is 2-torsion free, the above relation yields that $ty\Delta(x)y^*t^* = 0$ for all $x, y, t \in R$. Hence, the application of Lemma 5.3.1, yields that $y\Delta(x)y^* = 0$ for all $x, y \in R$. Again in view of Lemma 5.3.1, we are forced to conclude that $\Delta(x) = 0$ for all $x \in R$. That is, $D(x^2, z) - D(x, z)x^* - xD(x, z) = 0$ for all $x, z \in R$. That is, $D(x^2, z) = D(x, z)x^* + xD(x, z)$ for all $x, z \in R$. Hence, D is a symmetric Jordan $*$-biderivation on R. Thereby the proof is completed. □

5.4 Applications

In this section, we present some applications of our previous results obtained in Section 5.2. In [41], Brešar showed that if a prime ring R admits a nonzero derivation d such that $d(x)x - xd(x) \in Z(R)$ for all $x \in U$ or $d(x)x + xd(x) \in Z(R)$ for all $x \in U$, where U is a nonzero left ideal of R, then R is commutative. Further, the above mentioned result was extended by Argaç [14] as follows: Let R be a semiprime ring and d, g are derivations of R such that at least one is nonzero. If $d(x)x = xg(x)$ for all $x \in R$, then R has a nonzero central ideal. Moreover, if R is prime, then R is commutative. In the present section, our aim is to study similar problems in the setting of rings with involution involving pair of Jordan $*$-derivations d and g of R. We begin with the following theorem.

Theorem 5.4.1. *Let R be a noncommutative prime ring with involution $*$. Suppose there exist Jordan $*$-derivations $d, g : R \to R$ such that either $xd(y) - yg(x) = 0$ for all $x, y \in R$ or $xd(y) + yd(x) = 0$ for all $x, y \in R$. Then, $d = g = 0$.*

Proof. First we consider the case

$$xd(y) - yg(x) = 0 \text{ for all } x, y \in R. \tag{5.4.1}$$

Replacing y by y^2 in (5.4.1), we get $0 = xd(y^2) - y^2g(x) = xd(y)y^* + xyd(y) - y^2g(x)$ for all $x, y \in R$. Using the given hypothesis we obtain $xd(y)y^* + xyd(y) - yxd(y) = 0$

for all $x, y \in R$. That is,

$$xd(y)y^* + [x,y]d(y) = 0 \text{ for all } x,y \in R. \tag{5.4.2}$$

Replacing x by zx in (5.4.2), we get

$$zxd(y)y^* + z[x,y]d(y) + [z,y]xd(y) = 0 \text{ for all } x,y,z \in R. \tag{5.4.3}$$

Left multiplying (5.4.2) by z, yields

$$zxd(y)y^* + z[x,y]d(y) = 0 \text{ for all } x,y,z \in R. \tag{5.4.4}$$

Comparing (5.4.3) and (5.4.4), we obtain

$$[z,y]xd(y) = 0 \text{ for all } x,y,z \in R. \tag{5.4.5}$$

Thus for each $y \in R$, by the primeness of R either $[z,y] = 0$ or $d(y) = 0$. Now, let $A = \{y \in R | [z,y] = 0 \text{ for all } z \in R\}$ and $B = \{y \in R | d(y) = 0\}$. Then A and B are additive subgroups of R and $R = A \cup B$. But a group can not be a union of two of its proper subgroups and hence either $R = A$ or $R = B$. Since we have assumed R to be noncommutative. So we are forced to conclude that $R = B$. That is, $d(y) = 0$ for all $y \in R$. This intern implies $g = 0$ by (5.4.1). □

The same argument can be adopted in the case $xd(y) + yg(x) = 0$ for all $x, y \in R$.

Theorem 5.4.2. *Let R be a noncommutative prime ring with involution $*$ such that $char(R) \neq 2$. Suppose there exist Jordan $*$-derivations $d, g : R \to R$ such that either $xd(x) - xg(x) = 0$ for all $x \in R$ or $xd(x) + xg(x) = 0$ for all $x \in R$. Moreover, if $S(R) \cap Z(R) \neq (0)$, then $d = g = 0$.*

Proof. First we consider the case

$$xd(x) - xg(x) = 0 \text{ for all } x \in R. \tag{5.4.6}$$

Linearizing (5.4.6), we get

$$yd(x) + xd(y) = yg(x) + xg(y) \text{ for all } x,y \in R.$$

This can be further written as

$$y(d-g)(x)-x(g-d)(y)=0 \text{ for all } x,y\in R.$$

Now if d and g are Jordan $*$-derivations on R, then $d-g$ and $g-d$ are also Jordan $*$-derivations on R. Therefore in view of Theorem 5.4.1, we conclude that $d-g=0$ and $g-d=0$. That is, $d \doteq g$. In view of (5.4.6), we get $[d(x),x]=0$ for all $x\in R$ and $[g(x),x]=0$ for all $x\in R$. Since R is noncommutative prime ring with involution and $[d(x),x]=0$, so by Theorem 5.2.4, we are forced to conclude that $d=0$ and $g=0$. $\square$

The same argument can be adopted in the case $xd(x)+xg(x)=0$ for all $x\in R$.

Theorem 5.4.3. *Let R be a noncommutative prime ring with involution $*$ such that $char(R)\neq 2$. Suppose there exist Jordan $*$-derivations $d, g : R\to R$ such that either $d(x)y-yg(x)=0$ for all $x,y\in R$ or $d(x)y+yg(x)=0$ for all $x,y\in R$. Moreover, if $S(R)\cap Z(R)\neq(0)$, then $d=g=0$.*

Proof. First we consider the case

$$d(x)y-yg(x)=0 \text{ for all } x,y\in R. \tag{5.4.7}$$

Replacing y by yz in (5.4.7), we get

$$d(x)yz-yzg(x)=0 \text{ for all } x,y,z\in R. \tag{5.4.8}$$

Right multiplying (5.4.7) by z, we obtain

$$d(x)yz-yg(x)z=0 \text{ for all } x,y,z\in R. \tag{5.4.9}$$

Comparing (5.4.8) and (5.4.9), we get $y[g(x),z]=0$ for all $x,y,z\in R$. This further implies $[g(x),z]y[g(x),z]=0$ for all $x,y,z\in R$. Hence, the primeness of R forces that

$$[g(x),z]=0 \text{ for all } x,z\in R. \tag{5.4.10}$$

Replacing x by x^2 in (5.4.10), and using the fact g is a Jordan $*$-derivation, we get $0=[g(x^2),z]=[g(x)x^*+xg(x),z]=g(x)[x^*,z]+[x,z]g(x)$ for all $x,z\in R$. This further implies that $g(x)[x+x^*,z]=0$ for all $x,z\in R$. Replacing z by zy in the last expression, we obtain $g(x)z[x+x^*,y]=0$ for all $x,y,z\in R$. Thus for each $x\in R$, by the primeness of R either $g(x)=0$ or $[x+x^*,y]=0$. Now let $A=\{x\in R|g(x)=0\}$ and $B=\{x\in R|[x+x^*,y]=0$ for all $y\in R\}$. Thus A and B are additive subgroups of R and $R=A\cup B$. But a group can not be a union of two of its proper subgroups and hence either $R=A$ or $R=B$. Suppose $R=B$, then $[x+x^*,y]=0$ for all $x,y\in R$. Replacing x by $h+k$, where $h\in H(R)$ and $k\in S(R)$, we get $2[h,y]=0$. Since $char(R)\neq 2$, we obtain $[h,y]=0$ for all $h\in H(R)$ and $y\in R$. That is, $H(R)\subseteq Z(R)$. Thus R is commutative by Lemma 2.2.1. Which gives a contradiction. Therefore we must have $R=A$. That is, $g=0$. This intern implies that $d=0$ in view (5.4.7). $\square$

The same argument can be adopted in the case $d(x)y+yg(x)=0$ for all $x,y\in R$. Thereby completing the proof of the theorem.

Theorem 5.4.4. *Let m,n be fixed non-negative integers, and R be a $(m+n)!$-torsion free noncommutative prime ring with involution $*$ having the identity element e. Suppose there exist Jordan $*$-derivations $d,g:R\to R$ such that either $d(x^m)x^n-x^ng(x^m)=0$ for all $x\in R$ or $d(x^m)x^n+x^ng(x^m)=0$ for all $x\in R$. Moreover, if $S(R)\cap Z(R)\neq(0)$, then $d=g=0$.*

Proof. First we assume that

$$d(x^m)x^n-x^ng(x^m)=0 \text{ for all } x\in R. \tag{5.4.11}$$

Substituting $x+\lambda y$ for x in (5.4.11), we obtain

$$\lambda P_1(x,y)+\lambda^2P_2(x,y)+...+\lambda^{(m+n)}P_{(m+n)}(x,y)=0 \text{ for all } x,y\in R,$$

where $\lambda\in Z$ and $P_i(x,y)$ denotes the sum of terms involving i factors of y in the expansion of $d((x+\lambda y)^m)(x+\lambda y)^n-(x+\lambda y)^ng((x+\lambda y)^m)=0$. In view of Lemma 1.3.6, we obtain

$$\begin{aligned}P_1(x,y) &= d(x^{m-1}y+x^{m-2}yx+...+yx^{m-1})x^n+d(x^m)(x^{n-1}y+x^{n-2}yx+...+yx^{n-1})\\ &- (x^{n-1}y+x^{n-2}yx+...+yx^{n-1})g(x^m)-x^ng(x^{m-1}y+x^{m-2}yx+...+yx^{m-1})\end{aligned} \tag{5.4.12}$$

for all $x, y \in R$. Taking $x = e$ into (5.4.12) and noting that $d(e) = 0$ and $g(e) = 0$, we obtain

$$md(y) - mg(y) = 0 \text{ for all } y \in R. \tag{5.4.13}$$

Since R is $(m+n)!$-torsion free, we obtain $d(y) - g(y) = 0$ for all $y \in R$. That is,

$$d(y) = g(y) \text{ for all } y \in R. \tag{5.4.14}$$

Using a similar computational way to (5.4.12), we also have

$$P_2(e, y) = nmd(y)y + \frac{(m+1)m}{2}d(y^2) - nmyg(y) - \frac{(m+1)m}{2}g(y^2) \tag{5.4.15}$$

for all $y \in R$. Application of (5.4.14) yields that

$$nmd(y)y - nmyd(y) = 0 \text{ for all } y \in R.$$

Since R is $(m+n)!$-torsion free, we obtain $[d(y), y] = 0$ for all $y \in R$. Further in view of expression (5.4.14), we conclude that $[g(y), y] = 0$ for all $y \in R$. Since R is noncommutative prime ring, so by Theorem 5.2.4, we are forced to conclude that $d = 0$ and $g = 0$. $\square$

By the similar approach, we obtain the same conclusion in case $d(x^m)x^n + x^n g(x^m) = 0$ for all $x \in R$. This proves the theorem completely.

As an immediate consequence of Theorem 5.4.4, we have the following corollary.

Corollary 5.4.1. *Let R be a noncommutative prime ring with involution $*$ such that $char(R) \neq 2$, having the identity element e. Suppose there exist Jordan $*$-derivations $d, g : R \to R$ such that either $d(x)x - xg(x) = 0$ for all $x \in R$ or $d(x)x + xg(x) = 0$ for all $x \in R$. Moreover, if $S(R) \cap Z(R) \neq (0)$, then $d = g = 0$.*

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List of Publications

List of Published and Accepted Papers

1. Ali, Shakir, Dar, N. A., Vukman, J., Jordan left $*$-centralizers of prime and semiprime rings with involution, Beitr Algebra Geom 54 (2013), 609-624 DOI 10.1007/s13366-012-0117-3.

2. Ali, Shakir and Dar, N. A, On $*$-centralizing mappings in rings with involution, Georgian Math. J. 21(3) (2014). DOI 10.1515/gmj-2014-0006.

3. Ali, Shakir and Dar, N. A., On left centralizers of prime rings with involution, Plast. J. Math. 3(Specl 1) (2014) (To appear).

4. Ali, Shakir, Dar, N. A., Asci, M., On derivations and commutativity of prime rings with involution, Georgian Math. J. (2014) (accepted for publication).

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ORIGINAL PAPER

Jordan left $*$-centralizers of prime and semiprime rings with involution

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Abstract Let R be a ring with involution $'*'$. An additive mapping $T : R \to R$ is called a left $*$-centralizer (resp. Jordan left $*$-centralizer) if $T(xy) = T(x)y^*$ (resp. $T(x^2) = T(x)x^*$) holds for all $x, y \in R$, and a reverse left $*$-centralizer if $T(xy) = T(y)x^*$ holds for all $x, y \in R$. In the present paper, it is shown that every Jordan left $*$-centralizer on a semiprime ring with involution, of characteristic different from two is a reverse left $*$-centralizer. This result makes it possible to solve some functional equations in prime and semiprime rings with involution. Moreover, some more related results have also been discussed.

Keywords Prime ring · Semiprime ring · Involution · Left $*$-centralizer · Reverse left $*$-centralizer · Reverse $*$-centralizer · Jordan left $*$-centralizer

Mathematics Subject Classification (2000) 16N60 · 16W10

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Research Article

Shakir Ali and Nadeem Ahmed Dar

On $*$-centralizing mappings in rings with involution

Abstract: A classical result of Posner states that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. The main purpose of this paper is to prove a $*$-version of Posner's theorem mentioned above. Moreover, we describe the structure of an arbitrary additive mapping which is $*$-centralizing on a prime ring with involution.

Keywords: Prime ring, normal ring, involution, $*$-commuting mapping, $*$-centralizing mapping, derivation

MSC 2010: 16W10, 16N60, 16W25

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1 Introduction

Throughout this article, R will represent an associative ring with center $Z(R)$. We denote by Q_mr and C the maximal ring of quotients and the extended centroid of a prime ring R, respectively. For the explanation of Q_mr and C we refer the reader to [1]. A ring R is said to be 2-torsion free if $2a = 0$ (where $a \in R$) implies $a = 0$. A ring R is called a prime ring if $aRb = (0)$ (where $a, b \in R$) implies $a = 0$ or $b = 0$ and is called a semiprime ring in case $aRa = (0)$ implies $a = 0$. We write $[x, y]$ for $xy - yx$ and make extensive use of the basic commutator identities: $[xy, z] = x[y, z] + [x, z]y$, $[x, yz] = [x, y]z + y[x, z]$ for all $x, y, z \in R$. An additive map $x \mapsto x^*$ of R into itself is called an involution if (i) $(xy)^* = y^*x^*$ and (ii) $(x^*)^* = x$ hold for all $x, y \in R$. A ring equipped with an involution is known as a ring with involution or a $*$-ring. An element x in a ring with involution $*$ is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of R will be denoted by $H(R)$ and $S(R)$, respectively. If R is 2-torsion free, then every $x \in R$ can be uniquely represented in the form $2x = h + k$ where $h \in H(R)$ and $k \in S(R)$. Note that in this case x is normal, i.e., $xx^* = x^*x$, if and only if h and k commute. If all elements in R are normal, then R is called a normal ring. An example is the ring of quaternions. A description of such rings can be found in [5], where further references can be found.

An additive mapping $d : R \to R$ is said to be a derivation of R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. A derivation d is said to be inner if there exists $a \in R$ such that $d(x) = ax - xa$ for all $x \in R$. A mapping f of R into itself is called centralizing if $[f(x), x] \in Z(R)$ holds for all $x \in R$; in the special case when $[f(x), x] = 0$ holds for all $x \in R$, the mapping f is said to be commuting. The history of commuting and centralizing mappings goes back to 1995 when Divinsky [4] proved that a simple artinian ring is commutative if it has a commuting non-trivial automorphism. Two years later, Posner [7] proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Over the last few decades, several authors have proved commutativity theorems for prime and semiprime rings admitting automorphisms or derivations which are commuting or centralizing on an appropriate subset of the ring (see [2] for a partial bibliography).

Let R be a ring with involution $*$ and S be a nonempty subset of R. Motivated by the definition of centralizing mapping, the notion of $*$-centralizing and $*$-commuting mappings are defined as follows: a mapping f from R to R is called $*$-centralizing on S if $[f(x), x^*] \in Z(R)$ for all $x \in S$, and is called $*$-commuting on S if $[f(x), x^*] = 0$ for all $x \in S$. For any central element a, the map $x \mapsto ax^*$ is $*$-commuting and $*$-centralizing but not centralizing on R. Therefore it is reasonable to study these mappings in prime and semiprime rings with involution.

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Dear Prof. Shakir Ali

I am pleased to inform you that your paper titled " On left centralizers of prime rings with involution," has been accepted for publication in the Palestine J. Math.

This will appear very soon in special volume of Palestine J. Math 3(spec 1)(2014).

Ayman Badawi

Editor in chief of the PJM

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On left centralizers of prime rings with involution

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Abstract

The objective of the present paper is to prove that if a prime ring with involution of characteristic different from two admits a nonzero left centralizer T such that $[T(x), x^*] = 0$ for all $x \in R$, then R is normal. Further, we characterize normal rings and two sided centralizers among all prime rings with involution satisfying certain identities involving left centralizers.

2010 AMS Mathematics Subject Classification: 16N60, 16W10, 16W25.

Keywords and Phrases: Prime ring, normal ring, involution, left centralizer, centralizer, $*$-commuting mapping and $*$-centralizing mapping.

1. Introduction

This research is inspired by the work of Divinsky [8] and Vukman [24] . Throughout this article, R will represent an associative ring with centre $Z(R)$. We denote by $Q_l(R)$, $Q_m(R)$, $Q_s(R)$ and C, the maximal left ring of quotients, maximal right ring of quotients, the symmetric ring of Quotients and the extended centroid of a prime ring R. For the explanation of $Q_l(R)$, $Q_m(R)$, $Q_s(R)$ and C we refer the reader to [5]. A ring R is said to be 2-torsion free if $2a = 0$ (where $a \in R$) implies $a = 0$. A ring R is called a prime ring if $aRb = (0)$ (where $a, b \in R$) implies $a = 0$ or $b = 0$. We write $[x, y]$ for $xy - yx$ and make extensive use of basic commutator identities: $[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = [x, y]z + y[x, z]$ for all $x, y, z \in R$. An additive map $x \mapsto x^*$ of R into itself is called an involution if (i) $(xy)^* = y^*x^*$ and (ii) $(x^*)^* = x$ holds for all $x, y \in R$. A ring equipped with an involution is known as ring with involution or $*$-ring. An element x in

[0]This research is partially supported by the Research Grants (UGC No. 39-37/2010(SR)), *India*.

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ıs #GMJ/2013^208 is accepted for publication

message

eorgian Mathematical Journal <gmj@rmi.ge> Wed, Jun 11, 2014 at :

): Nadeem Dar <ndmdarlajurah@gmail.com>

Dear Professor Dar,

The referee is satisfied by the revised version of ms #GMJ/2013^208. Thus your ms is accepted for publicati GMJ.

Sincerely,

Maia Kvinikadze
Editorial Secretary, GMJ

> ---- Original Message ----
> **From:** Nadeem Dar
> **To:** Georgian Mathematical Journal
> **Sent:** Wednesday, June 11, 2014 8:08 AM
>
> Dear Prof. Kiguradze, Ivan
>
> How are you. Did you received the revised version of our paper with manuscript ID: #GMJ/2013^208
>
> Sincerely yours
> Nadeem Ahmad Dar
> Department of mathematics
> Aligarh Muslim University
> Aligarh –202002(U.P.)
> India

›port on ms #GMJ/2013^208

nessages

›orgian Mathematical Journal <gmj@rmi.ge> Thu, Jun 5, 2014 at 2
: Nadeem Dar <ndmdarlajurah@gmail.com>

Dear Professor Nadeem Ahmad Dar,

Please, find attached referee's report on your joint ms #GMJ/2013^208.
According to this report after taking into account referee's remars your ms will be accepted for publication in GMJ.

Sincerely,
Maia Kvinikadze
Editorial Secretary, GMJ

— Original Message — From: "Georgian Mathematical Journal" <gmj@rmi.ge>
To: "Nadeem Dar" <ndmdarlajurah@gmail.com>
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Sent: Monday, December 02, 2013 11:55 AM
Subject: ms #GMJ/2013^208 by S. Ali, N. A. Dar & M. Asci [28/11/13]

To: Prof. Nadeem Ahmad Dar

Your joint ms mentioned below is registered under #GMJ/2013^208
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Date: Thu, 28 Nov 2013 19:22:57 +0530
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To: kig@rmi.ge
Subject: Submission of Manscript

Dear Prof. Kiguradze, Ivan,

Editor

Georgian Mathematical Journal

Please find attached with the PDF file and the source file along with word
file containing the names of four possible referees for the possible
publication of our paper " On derivations and commutativity of prime rings

Georgian Math. J. ? (????), 1–11
DOI 10.1515/gmj-????-???

On derivations and commutativity of prime rings with involution

Shakir Ali, Nadeem Ahmed Dar and Mustafa Asci

Abstract. In [8], Bell and Daif proved if R is a prime ring admitting a nonzero derivation such that $d(xy) = d(yx)$ for all $x, y \in R$, then R is commutative. The objective of this paper is to examine the similar problems when the ring R is equipped with involution. It is shown that if a prime ring R with involution $*$ of characteristic different from 2 admits a nonzero derivation d such that $d(xx^*) = d(x^*x)$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$, then R is commutative. Moreover, some related results have also been discussed.

Keywords. Prime ring, normal ring, involution, derivation..

2010 Mathematics Subject Classification. 16W10, 16N60, 16W25..

1 Introduction

This research has been motivated by our earlier work [1]. Throughout this article, R will represent an associative ring with centre $Z(R)$. For $a, b \in R$, $[a, b]$ will be the element $ab - ba$ and $a \circ b$ the element $ab + ba$, respectively. However, given two subsets A and B of R, then $[A, B]$ will denote the additive subgroup of R generated by all elements of the form $[a, b]$ where $a \in A$ and $b \in B$ and $A \circ B$ is defined similarly. Further, $\overline{A}$ will be the subring of R generated by A. A ring R is said to be 2-torsion free if $2a = 0$ (where $a \in R$) implies $a = 0$. A ring R is called a prime ring if $aRb = (0)$ (where $a, b \in R$) implies $a = 0$ or $b = 0$, and is called a semiprime ring in case $aRa = (0)$ implies $a = 0$. An additive map $x \mapsto x^*$ of R into itself is called an involution if (i) $(xy)^* = y^*x^*$ and (ii) $(x^*)^* = x$ hold for all $x, y \in R$. A ring equipped with an involution is called a ring with involution or a $*$-ring. An element x in a ring with involution $*$ is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of R will be denoted by $H(R)$ and $S(R)$, respectively. If R is 2-torsion free then every $x \in R$ can be uniquely represented in the form $2x = h + k$ where $h \in H(R)$ and $k \in S(R)$. Note that in this case x is normal i.e., $xx^* = x^*x$, if and only if h and k commute. If all elements in R are normal, then R is called a

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ON CENTRALIZERS OF PRIME RINGS WITH INVOLUTION

SHAKIR ALI AND NADEEM AHMAD DAR

Communicated by

ABSTRACT. Let R be a ring with involution '$*$'. An additive mapping $T : R \to R$ is called a left(resp. right) centralizer if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) for all $x, y \in R$. The purpose of this paper is to examine the commutativity of prime rings with involution satisfying certain identities involving left centralizers.

1. Introduction

Throughout this article, R will represent an associative ring with centre $Z(R)$. A ring R is said to be 2-torsion free if $2a = 0$ (where $a \in R$) implies $a = 0$. A ring R is called a prime ring if $aRb = (0)$ (where $a, b \in R$) implies $a = 0$ or $b = 0$. We write $[x, y]$ for $xy - yx$ and xoy for $xy + yx$, respectively. An additive map $x \mapsto x^*$ of R into itself is called an involution if (i) $(xy)^* = y^*x^*$ and (ii) $(x^*)^* = x$ holds for all $x, y \in R$. A ring equipped with an involution is known as ring with involution or $*$-ring. An element x in a ring with involution '$*$' is said to hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of R will be denoted by $H(R)$ and $S(R)$, respectively. If R is 2-torsion free then every $x \in R$ can be uniquely represented in the form $2x = h + k$, where $h \in H(R)$ and $k \in S(R)$.

MSC(2010): Primary: 16N60; Secondary: 16W10.

Keywords: Prime ring, normal ring, involution, left centralizer, centralizer.

This research is partially supported by a Major Research Project funded by U.G.C. (Grant No. 39-37/2010(SR))

Subject: Re: 461

From: "umzh" <umzh@imath.kiev.ua> Wed, 27 Nov '13 2:30p

To: You Show full Headers | View blocked images

Dear Dr. Shakir Ali,.

This is in reference to your submitted paper entitled " "A characterization of additive mappings in rings with involution " jointly with N. A. DAR.

I am glad to inform you that our reviewer has positive opinion about your work and has few comments for revision. I am happy to accept it for publication in our Journal (Ukrainian Mathematical Journal).

Best regards,

T. G. Mitseruk (editor)

Ukrainian Mathematical Journal

A characterization of additive mappings in rings with involution

Shakir Ali* and Nadeem Ahmad Dar

ABSTRACT: The main purpose of this paper is to characterize some additive mappings satisfying certain functional equations in rings with involution. In particular, we prove that any Jordan $*$-centralizer on a 2-torsion free semiprime $*$-ring is a reverse $*$-centralizer. Further, we establish that if R is a $(m+n)!$ torsion free noncommutative prime ring with involution $*$ and D, G are Jordan $*$-derivations on R such that $D(x^m)x^n \pm x^n G(x^m) = 0$ for all $x \in R$, where m, n are non-negative integers, then $D = G = 0$. This result is in the spirit of the classical result of E. C. Posner [26], which states that: Let R be a prime ring and D a derivation of R such that $xD(x) - D(x)x = 0$ for all $x \in R$. Then R is commutative or $D = 0$.

Keywords: Prime ring, semiprime ring, involution, $*$-derivation, Jordan $*$-derivation, reverse $*$-centralizer, Jordan $*$-centralizer.

2000 Mathematics Subject Classification: 16W10, 16N60, 16W25.

1. Introduction

Throughout this paper, R will represent an associative ring with centre $Z(R)$. We denote by Q_mr, Q_r, Q_s and C the maximal ring of quotients, the right Martindale ring of quotients, the symmetric Martindale ring of quotients and the extended centroid of R. For the explanation of Q_mr, Q_r, Q_s and C, we refer the reader to [7]. For all $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol

[0] *This research is partially supported by the research grants from UGC, India (Grant No. 39-37/2010(SR), India)*

* *Corresponding author*